D. Hui
Doctoral Research Assistant.
Assoc. Mem. ASME

J. S. Hansen
Assistant Professor.
Institute of Aerospace Studies,
University of Toronto,
Downsview,
Ontario, Canada M3H 5T6

Two-Mode Buckling of an Elastically Supported Plate and Its Relation to Catastrophe Theory

Classical buckling and initial postbuckling of a geometrically imperfect infinite plate on a nonlinear elastic foundation under two independent applied compressive loads are analyzed. The plate is assumed to have imperfections of the same form as the buckling modes. It is found that single mode behavior occurs when the two independent loads N_x and N_y are unequal. A two-mode case occurs when the two applied loads are equal and the form of the instability falls into the category of the parabolic umbilic type one or type two, depending on the quadratic and cubic spring constants. The importance of the contribution of the quartic term and imperfection-sensitivity is examined. The analysis is studied within the context of Koiter's general theory of multimode postbuckling behavior.

Introduction

Catastrophe theory [1] and the theory of elastic stability [2–5] are two independently developed theories which deal with the stability analysis of physical systems which evolve as a function of certain prescribed parameters. As such, these theories are closely related to one another and have resulted in a number of papers [6–8] which provide comparative studies. It is evident from the foregoing that catastrophe theory is effective in the classification of the forms of instability which may occur while the task of actually providing a method of analysis of physical systems has been taken up in the theory of elastic stability.

Thompson and Hunt [6] were among the first to investigate the similarity between the theory of elastic stability and catastrophe theory and they provided a comparative study of the various types of instability mechanisms. Sewell [7] provided a series of examples which demonstrated various forms of the elementary catastrophes while Huseyin [8] considered the comparison between the theory of multiple-parameter systems and catastrophe theory. Further, a general analysis of two-mode buckling problems and their relation to the hyperbolic and elliptic catastrophes were presented in [9]. In addition, the parabolic umbilic catastrophe was first analyzed in depth in terms of the theory of elastic stability in [10] and was then applied to the two-mode buckling problem of an imperfection-sensitive externally pressurized spherical shell. Further applications to various simple structures can also be found in [11, 12].

Contributed by the Applied Mechanics Division for publication in the JOURNAL OF APPLIED MECHANICS.

Discussion on this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N. Y. 10017, and will be accepted until December 1, 1980. Readers who need more time to prepare a discussion should request an extension from the Editorial Department. Manuscript received by ASME Applied Mechanics Division, February, 1979; final revision, October, 1979.

The present paper deals with the two-mode initial postbuckling analysis of an infinite plate resting on a nonlinear elastic foundation. It is found that the resulting form of the potential energy falls into the category of the parabolic umbilic model of catastrophe theory. This represents an extension of a paper by Reissner [13] in which he showed that this problem is qualitatively similar to the two-mode buckling problem of an externally pressurized spherical shell [14] in that the expanded potential energy takes a similar form. A Koiter style analysis is used and the problem is attacked using a UVW displacement formulation.

The analysis considers the case of two independent compressive in-plane loads N_x and N_y and it is shown that in general there exists a unique eigenvalue for the buckling problem. However, in the particular situation that $N_x = N_y$, an infinite number of buckling modes are involved. In the present case, the analysis is focused on a two-mode interaction problem and it is shown that this leads to a stability problem which takes the form of the parabolic umbilic catastrophe. Critical load-imperfection results are obtained and it is demonstrated that the inclusion of two independent load parameters, as specified by catastrophe theory, can significantly alter the results. In addition, the results show that higher-order terms of the potential energy for the present two-mode plate buckling problem cannot always be neglected in this type of asymptotic analysis.

Potential Energy

The potential energy of a plate resting on a nonlinear elastic foundation can be expressed as

$$\text{P.E.} = U_m + U_b + U_F - U_W \qquad (1)$$

where U_m is the membrane strain energy, U_b is the bending strain energy, U_F is the strain energy of the elastic foundation, and U_W is the work done (positive for compessive loads) by the applied load. The aforementioned quantities are given by

$$U_m = \frac{Et}{2(1-\nu^2)} \iint \left\{ \left[U_{,x} + \frac{1}{2} W^2{}_{,x} + V_{,y} + \frac{1}{2} W^2{}_{,y} \right]^2 - 2(1-\nu) \left[\left(U_{,x} + \frac{1}{2} W^2{}_{,x} \right) \left(V_{,y} + \frac{1}{2} W^2{}_{,y} \right) - \frac{1}{4} \left(U_{,y} + V_{,x} + W_{,x} W_{,y} \right)^2 \right] \right\} dxdy \quad (2)$$

$$U_b = \frac{Et^3}{24(1-\nu^2)} \iint \left[\left(W_{,xx} + W_{,yy} \right)^2 - 2(1-\nu)\left(W_{,xx} W_{,yy} - W^2{}_{,xy} \right) \right] dxdy \quad (3)$$

$$U_F = \frac{Et}{2(1-\nu^2)} \iint \left[\frac{1}{2} k_1 W^2 + \frac{1}{3} k_2 W^3 + \frac{1}{4} k_3 W^4 \right] dxdy \quad (4)$$

and

$$U_W = \int\int N_x U_{,x} dxdy + \int\int N_y V_{,y} dxdy \quad (5)$$

In the foregoing, E is Young's modulus, t is the thickness of the plate, ν is Poisson's ratio, (U, V, W) is the displacement vector of the middle plane of the plate, k_2, k_2, k_3 are related to the linear, quadratic, and the cubic spring constants of the elastic foundation, respectively, and N_x, N_y are the in-plane, applied loads in the x,y-directions respectively.

The prebuckling state of a plate is composed of end shortening in the two in-plane directions with no out-of-plane displacement. Therefore, the total displacement takes the form

$$U = c_1 x + u, \quad V = c_2 y + v, \quad W = w \quad (6)$$

where c_1 and c_2 are functions of the applied loads and u, v, w are incremental displacements which are zero prior to buckling.

Substituting the total displacement into the potential energy expression and then grouping the terms according to the powers of the incremental displacements, the potential energy may be expressed as

$$P^\lambda[u] = P_1{}^\lambda[u] + P_2{}^\lambda[u] + P_3{}^\lambda[u] + P_4{}^0[u] + \ldots \quad (7)$$

where $P_i{}^\lambda[u]$ is a function of the ith degree in the perturbed displacements and u represents the vector of displacements $[u, v, w]^T$. The superscript λ indicates that the functional contains terms which depend on the applied loads and the superscript 0 implies that the functional is independent of the applied loads.

Explicit values for c_1 and c_2 are obtained from the requirement that $P_1{}^\lambda[u]$ must vanish in order that the prebuckling state be an equilibrium state. Using this condition yields

$$c_1 = -\frac{1}{Et}[N_x - \nu N_y]; \quad c_2 = -\frac{1}{Et}[N_y - \nu N_x] \quad (8)$$

The remaining quantities $P_2{}^0[u]$, $P_2{}'[u]$, $P_2{}^*[u]$, and $P_4{}^0[u]$ are

$$P_2{}^0[u] = \frac{Et}{2(1-\nu^2)} \iint \left\{ u^2{}_{,x} + v^2{}_{,y} + \frac{1}{2}(1-\nu)(u_{,y} + v_{,x})^2 + 2\nu(u_{,x} v_{,y}) + \left(\frac{t^2}{12}\right) [w^2{}_{,xx} + w^2{}_{,yy} + 2\nu w_{,xx} w_{,yy} + 2(1-\nu) w^2{}_{,xy}] + \frac{1}{2} k_1 w^2 \right\} dxdy \quad (9)$$

$$P_2{}'[u] = -\iint \frac{1}{2} w^2{}_{,x} dxdy \quad (10)$$

$$P_2{}^*[u] = -\iint \frac{1}{2} w^2{}_{,y} dxdy \quad (11)$$

$$P_3{}^0[u] = \frac{Et}{2(1-\nu^2)} \iint u_{,x} w^2{}_{,x} + v_{,y} w^2{}_{,y} + \nu(u_{,x} w^2{}_{,y}) - (v_{,y} w^2{}_{,x}) + (1-\nu)(u_{,y} + v_{,x})(w_{,x} w_{,y}) + \frac{1}{3} k_2 w^3 dxdy \quad (12)$$

$$P_4{}^0[u] = \frac{Et}{2(1-\nu^2)} \iint \frac{1}{4} (w^2{}_{,x} + w^2{}_{,y})^2 + \frac{1}{4} k_3 w^4 dxdy \quad (13)$$

where

$$P_2{}'[u] = \frac{\partial}{\partial N_x} P_2{}^\lambda[u] \quad \text{and} \quad P_2{}^*[u] = \frac{\partial}{\partial N_y} P_2{}^\lambda[u]$$

In the foregoing, it may be noted that $P_3{}^0[u]$ has replaced $P_3{}^\lambda[u]$. This results because of the linearity of the prebuckling problem.

Classical Buckling Load

The classical critical load is determined from the condition that the first and second variations of the quadratic terms in the potential energy must vanish. These calculations take the form of an eigenvalue problem in terms of N_x and N_y, with the set of minimum values corresponding to the locus of classical critical loads.

The eigenfunctions are easily found as

$$u_{k_x k_y} = 0; \quad v_{k_x k_y} = 0$$

$$w_{k_x k_y} = \frac{\sin}{\cos}\left(k_x \frac{x}{q}\right) \frac{\sin}{\cos}\left(k_y \frac{y}{q}\right)$$

with the corresponding eigenvalues defined by

$$\lambda_x k_x{}^2 + \lambda_y k_y{}^2 = \frac{1}{2}[(k_x{}^2 + k_y{}^2)^2 + 1] \quad (14)$$

In obtaining these results the nondimensional quantities λ_x, λ_y have been introduced as

$$(\lambda_x, \lambda_y) = \frac{6(1-\nu^2)q^2}{Et^3}(N_x, N_y)$$

where $q = [t^2/6k_1]^{1/4}$. In addition, k_x, k_y are the wave numbers in the x, y-directions, respectively.

The classical critical load is obtained as the least value of this expression when it is minimized with respect to the wave numbers $k_x{}^2$ and $k_y{}^2$. Also, since λ_x and λ_y are assumed to be independent, there is a locus of values of λ_x and λ_y which defines the classical critical load. This locus of values may be determined in a number of ways; however, in the present case it is determined by assuming a prescribed relationship between λ_x and λ_y and another parameter λ, and then determining the least value of λ. Doing so yields

$$\lambda_x \equiv \alpha\lambda; \quad \lambda_y \equiv \beta\lambda$$

where α and β (not both zero) take the values $0 \leq (\alpha, \beta) \leq 1$. Thus the eigenvalue equation can be rewritten as

$$\lambda = \frac{(k_x{}^2 + k_y{}^2)^2 + 1}{2(\alpha k_x{}^2 + \beta k_y{}^2)} \quad (15)$$

For the case $0 \leq \beta < \alpha$ it may be shown that the minimum eigenvalues are

$$\lambda_{x_{cl}} = 1; \quad 0 \leq \lambda_{y_{cl}} < 1 \quad (16)$$

where

$$\lambda_{y_{cl}} = \left(\frac{\beta}{\alpha}\right)$$

corresponding to the wave numbers

$$k_x{}^2 = 1; \quad k_y{}^2 = 0 \quad (17)$$

A second possibility $0 \leq \alpha < \beta$ follows in a parallel manner and yields identical results with the interchange of $\lambda_{x_{cl}}$ and $\lambda_{y_{cl}}$ as well as k_x and k_y in the aforementioned. The third possibility which arises is that due to the uniform compression case $\alpha = \beta$. For this situation the least eigenvalue is

$$\lambda_{x_{cl}} = 1; \quad \lambda_{y_{cl}} = 1 \quad (18)$$

corresponding to the critical wave numbers

$$k_x{}^2 + k_y{}^2 = 1 \quad (19)$$

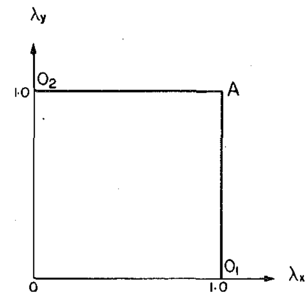

Fig. 1 Interaction of the critical loads $\lambda_{x_{cl}}$ and $\lambda_{y_{cl}}$

The previous eigenvalue will be designated as λ_{cl} or in dimensional form

$$N_{cl} = \frac{Et^3}{6(1-\nu^2)q^2}$$

These results, which yield the buckling load interaction for the independent applied loads N_x and N_y, are presented in Fig. 1. The interaction curves are obtained as the straight lines O_1A and O_2A, respectively. Excluding the point A, any point which lies on the line O_1A implies a state of single mode buckling. Similarly, any point on the line O_2A leads to a state of single mode buckling. The point A represents the case when the applied loads are of equal magnitude and for which multimode buckling occurs as there exist more than one set of wave numbers k_x and k_y which satisfy the condition for a minimum eigenvalue.

From this point on, the analysis will concentrate on the two-mode buckling problem which results when the critical wave numbers are given by

$$k_x = 1, \quad k_y = 0 \quad \text{First mode}$$
$$k_x = \frac{1}{2}, \quad k_y = \frac{\sqrt{3}}{2} \quad \text{Second mode} \tag{20}$$

or by the identical situation when k_x and k_y are interchanged. The specification of this particular combination of k_x and k_y may seem rather arbitrary and quite restricitve; however, it represents an important stepping stone to the multiple mode buckling situations. That is, for the possibilities involving $\lambda_{x_{cl}} \neq \lambda_{y_{cl}}$, single mode buckling occurs which has been adequately treated elsewhere. In addition, for the situation $\lambda_{x_{cl}} = \lambda_{y_{cl}}$, the foregoing combinations of k_x and k_y lead to the only two-mode problem which has nontrivial cubic terms in the potential energy. Thus, since higher-order instabilities contain the lower-order ones as special cases, it is appropriate to obtain a complete understanding of the lower-order problem. This has not been considered previously.

Initial Postbuckling of a Two-Mode System

Following Koiter's method of analysis [2], the potential energy of a two-mode system can be expanded in a Taylor's series about the classical critical load of the perfect system. Upon expansion, the approximation to the potential energy becomes

$$PE = (N_x - N_{cl})P_2'[u_c] + (N_y - N_{cl})P_2{}^*[u_c] + P_3{}^0[u_c] + \ldots + N_xP_{11}'[u_c, \bar{u}] + N_yP_{11}{}^*[u_c, \bar{u}] + \ldots \tag{21}$$

where $\bar{u}$ are initial imperfections which are taken in the same form as the buckling modes. The two sets of wave numbers given in the last section yields the critical modes

$$u_c{}^1 = [u_c{}^1, v_c{}^1, w_c{}^1]^T = t\xi_1[0, 0, \cos(x/q)]^T$$
$$u_c{}^2 = [u_c{}^2, v_c{}^2, w_c{}^2]^T = t\xi_2[0, 0, \cos(x/2q)\cos(\sqrt{3}y/2q)]^T \tag{22}$$

Upon substitution of the eigenvectors into the potential energy and carrying out the appropriate integration, the quadratic terms become

$$P_2'[u_c] = -\frac{1}{4}\left(\xi_1{}^2 + \frac{\xi_2{}^2}{8}\right)S_0\left(\frac{t}{q}\right)^2$$
$$P_2{}^*[u_c] = -\frac{1}{4}\left(\frac{3}{8}\xi_2{}^2\right)S_0\left(\frac{t}{q}\right)^2 \tag{23}$$

The terms involving the initial imperfections are

$$P_{11}'[u_c, \bar{u}] = -\frac{1}{2}\left(\xi_1\bar{\xi}_1 + \frac{1}{8}\xi_2\bar{\xi}_2\right)S_0\left(\frac{t}{q}\right)^2$$
$$P_{11}{}^*[u_c, \bar{u}] = -\frac{1}{2}\left(\frac{3}{8}\xi_2\bar{\xi}_2\right)S_0\left(\frac{t}{q}\right)^2 \tag{24}$$

where $\bar{\xi}_1\bar{\xi}_2$ are the imperfection amplitudes. The cubic contribution is given by

$$P_3{}^0[u_c] = \frac{Et}{2(1-\nu^2)}\iint \frac{1}{3}k_2(w_c{}^1 + w_c{}^2)^3 dx\,dy,$$

which can be expressed as

$$P_3{}^0[u_c] = P_3{}^0[u_c{}^1] + P_{21}{}^0[u_c{}^1, u_c{}^2] + P_{12}{}^0[u_c{}^1, u_c{}^2] + P_3{}^0[u_c{}^2]$$

Upon evaluation, the various terms become

$$P_{12}{}^0[u_c{}^1, u_c{}^2] = \frac{Et}{2(1-\nu^2)}(\xi_1\xi_2{}^2)(t^3k_2/8)S_0$$
$$P_3{}^0[u_c{}^1] = P_{21}{}^0[u_c{}^1, u_c{}^2] = P_3{}^0[u_c{}^2] = 0 \tag{25}$$

The fact that the last three terms vanish leads to the requirement [10] that higher-order quantities must be retained in the first approximation to the potential energy. The appropriate quantity for the present problem is a quartic and with the inclusion of this term the problem falls into the classification of the parabolic umbilic catastrophe. The additional term which is required is $P_4{}^0[u_c{}^1] - P_2{}^\lambda[u_2]$ where

$$P_4{}^0[u_c{}^1] = \frac{Et}{2(1-\nu^2)}\left[\frac{3}{32}\right][1 + q^4k_3]\left[\frac{t}{q}\right]^4 \xi_1{}^4S_0 \tag{26}$$

and where $u_2 = [u_2, v_2, w_2]^T$ is the solution of the second-order perturbation problem. Following Koiter [2], u_2 is given by the solution of

$$P_{11}{}^\lambda[u_2, \delta u_2] = -P_{21}{}^0[u_c{}^1, \delta u_2] \tag{27}$$

subject to the appropriate orthogonality conditions for u_2. The set of differential equations for u_2, v_2, w_2 is then obtained as

$$2u_{2,xx} + (1+\nu)v_{2,xy} + (1-\nu)u_{2,yy} = -\xi_1{}^2\frac{t^2}{q^3}\sin\left(\frac{2x}{q}\right)$$
$$2v_{2,yy} + (1+\nu)u_{2,xy} + (1-\nu)v_{2,xx} = 0 \tag{28}$$
$$\frac{t^2}{6}[w_{2,xxxx} + 2w_{2,xxyy} + w_{2,yyyy}] + k_1w_2 + \frac{2(1-\nu^2)}{Et}[N_xw_{2,xx} + N_yw_{2,yy}] = -\frac{1}{2}\xi_1{}^2t^2k_2\left(1 + \cos\frac{2x}{q}\right)$$

and where it is noted that the influence of the boundary conditions has been omitted and are replaced by a periodicity requirement.

The solution of the aforementioned differential equations is of the form

$$u_2 = (s_1)(\xi_1{}^2)\left(\sin\frac{2x}{q}\right)$$
$$v_2 = s_2$$
$$w_2 = s_3\xi_1{}^2 + s_4\xi_1{}^2\cos\frac{2x}{q} \tag{29}$$

Since only derivatives of v_2 appear in $P_2{}^\lambda[u_2]$ then s_2 need not be evaluated. Thus, substituting the expressions for u_2 and w_2 into the differential equations, the constants s_1, s_3, and s_4 can be evaluated. Doing so yields

$$s_1 = \frac{t^2}{8q}, \quad s_3 = -\frac{t^2}{2}\frac{k_2}{k_1}, \quad s_4 = -\frac{t^2}{18}\frac{k_2}{k_1}$$

and therefore the desired modification to the quartic term is

$$-P_2{}^\lambda[u_2] = \frac{Et}{2(1-\nu^2)}\left(\frac{t}{q}\right)^4\left\{-\frac{1}{32} - \frac{19q^4}{144}\frac{(k_2)^2}{k_1}\right\}\xi_1{}^4 S_0 \qquad (30)$$

Now, assembling the results from equations (23)–(26) and (30), the approximation to the potential energy becomes

$$PE = \frac{Et}{2(1-\nu^2)}\left(\frac{t}{q}\right)^4 S_0\left\{\frac{1}{12}\left(1-\lambda_x\right)\left(\xi_1{}^2 + \frac{1}{8}\xi_2{}^2\right)\right.$$
$$+ \frac{1}{12}(1-\lambda_y)\frac{3}{8}\xi_2{}^2 + \frac{1}{8}K_2\xi_1\xi_2{}^2 + C_{40}\xi_1{}^4$$
$$-\frac{1}{6}\lambda_x(\bar{\xi}_1\xi_1 + \frac{1}{8}\bar{\xi}_2\xi_2)$$
$$\left. -\frac{1}{6}\lambda_y\left(\frac{3}{8}\bar{\xi}_2\xi_2\right)\right\} \qquad (31)$$

where

$$C_{40} = \frac{1}{16}\left[1 + \frac{3}{2}K_3 - \frac{38}{3}(K_2)^2\right] \qquad (32)$$

and where $K_2 = (q^4/t)\,k_2$, $K_3 = q^4k_3$ are nondimensional spring constants. The equilibrium equations and stability determinant for the foregoing may now be obtained directly as

$$(1-\lambda_x)\xi_1 + \frac{3}{4}K_2\xi_2{}^2 + 24C_{40}\xi_1{}^3 = \lambda_x\bar{\xi}_1$$
$$\left[\frac{1}{8}(1-\lambda_x) + \frac{3}{8}(1-\lambda_y)\right]\xi_2 + \frac{3}{2}K_2\xi_1\xi_2$$
$$= \frac{1}{8}(\lambda_x + 3\lambda_y)\bar{\xi}_2 \qquad (33)$$

and

$$\begin{bmatrix} (1-\lambda_x) + 72C_{40}\xi_1{}^2 & \frac{3}{2}K_2\xi_2 \\ \frac{3}{2}K_2\xi_2 & \left[\frac{1}{8}(1-\lambda_x) + \frac{3}{8}(1-\lambda_y)\right. \\ & \left. + \frac{3}{2}K_2\xi_1\right] \end{bmatrix} \qquad (34)$$

respectively.

Transformation to Standard Form

The previously obtained potential energy expression can be transformed to the standard form of the parabolic umbilic by introducing the nondimensional quantities x, y, L_1, L_2, ϵ_1, and ϵ_2 (x and y not to be confused with coordinates of plate). The appropriate expressions are

$$x = C^*\xi_1, \quad y = [K_2/8C^*]^{1/2}\xi_2$$

$$L_1 = \frac{1}{12}(1-\lambda_x)/C^{*2} \quad L_2 = \frac{1}{12}[(1-\lambda_x) + 3(1-\lambda_y)][C^*/K_2]$$

$$\epsilon_1 = \frac{1}{6}[\lambda_x/C^*]\bar{\xi}_1 \quad \epsilon_2 = \frac{1}{6}(\lambda_x + 3\lambda_y)\sqrt{C^*/8K_2}\,\bar{\xi}_2$$

where

$$C^* = \sqrt[4]{|C_{40}|}.$$

Thus, after division by a constant, the potential energy is transformed to the form

$$PE = \pm x^4 + xy^2 + L_1x^2 + L_2y^2 - \epsilon_1 x - \epsilon_2 y$$

where the "+" or "−" signs correspond to C_{40} being positive or negative, respectively. The stability problem is therefore characterized by four control parameters (L_1, L_2, ϵ_1, ϵ_2) which are defined by the loads λ_x and λ_y and the initial geometric imperfections $\bar{\xi}_1$ and $\bar{\xi}_2$. The two cases defined by a positive or negative coefficient C_{40} result in distinctive stability problems and these are termed the parabolic umbilic types one and two, respectively. Fig. 2 shows the range of parameters K_2, K_3 which lead to either of these possibilities. Here, the hatched region corresponds to the type one case, the line represents a singular case when C_{40} vanishes (which is not treated in the context of the present analysis) and the remainder is the type two case.

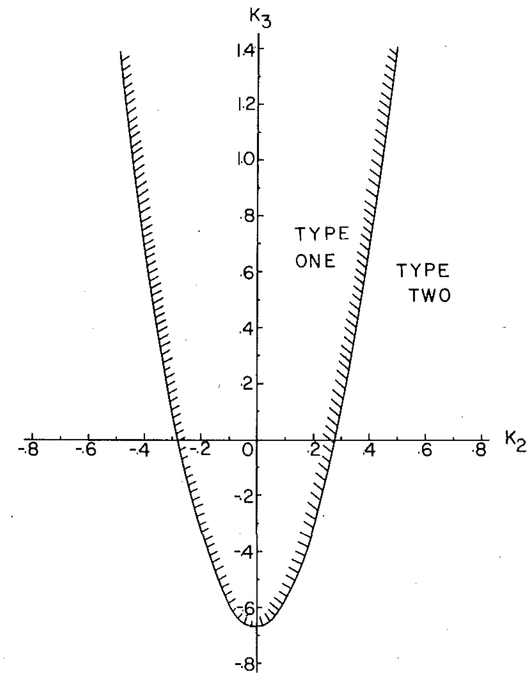

Fig. 2 Range of spring constants K_2 and K_3 which leads to the parabolic umbilic type one and type two models

Results

In order to show the influence of the quartic term as well as that of independent loads a series of representative curves demonstrating these parameters has been evaluated. The results are for the particular two-mode case corresponding to the modes of equation (22) and in the form of selected critical load-initial imperfection curves resulting from the stability problem described by equations (33) and (34). For the purposes of illustration K_2 and K_3 have been chosen as $K_2 = 0.133$, $K_3 = -0.4375$ and -0.5955 which yield $3/4\ K_2 = 0.1$ and $24C_{40} = \pm 0.177$ and where the type one and type two cases occur for positive and negative values of C_{40}, respectively. Further, for comparison purposes, in Fig. 3 the results for $C_{40} = 0$ are also presented. It is felt that the foregoing choice of coefficients yields the possibility of obtaining an unbiased parameter study for the present problem.

Fig. 3 demonstrates the differences in imperfection sensitivity for the type one and type two cases and a further comparison curve when the quartic term is omitted is also evaluated. It is noted that the applied loads are constrained to be equal ($\lambda_x = \lambda_y$). Of importance in this figure are the changes in imperfection sensitivity even for small imperfections and perhaps more importantly the existence of a critical load curve only in the type two case for positive values of the imperfection. Thus, if the quartic term is omitted in the analysis, there are fundamental changes in the critical load behavior of the type two case, even in an asymptotic sense.

Figs. 4 and 5 are denoted to an evaluation of the applied loads, λ_x and λ_y, being independent. This feature manifests itself in the factor ϵ. For the situation presented in Fig. 4 it may be appreciated that there

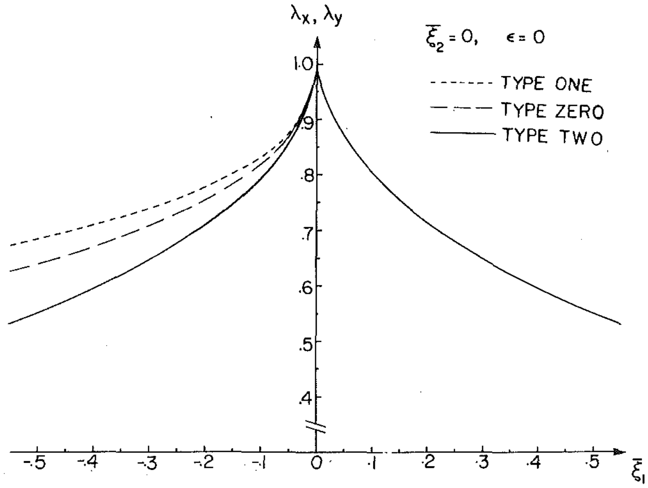

Fig. 3 Critical load-imperfection curves for equal applied loads ($\lambda_x = \lambda_y$) and $\bar{\xi}_2 = 0$

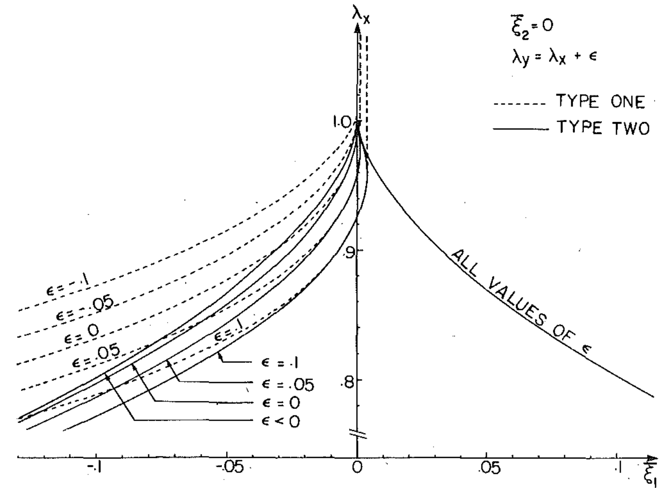

Fig. 4 Critical load-imperfection curves for unequal applied loads ($\lambda_y = \lambda_x + \epsilon$) and $\bar{\xi}_2 = 0$

are indeed changes in the critical loads as a function of ϵ, this change being quite uniform. The most interesting aspect is that for ϵ positive the critical load curve intersects the vertical axis. This implies a decrease in $\lambda_{x_{cr}}$ even for $\bar{\xi}_1 = 0$ and in addition that the type one situation becomes imperfection sensitive to positive $\bar{\xi}_1$ which was not the case for equal applied loads ($\lambda_x = \lambda_y$). Thus, although it is true for only very small positive $\bar{\xi}_1$, there has been a fundamental asymptotic change in the critical load-initial imperfection result. It is further noted in Fig. 4 that the type one and type two cases yield asymptotically similar results for negative $\bar{\xi}_1$ although the results do show different trends even for small imperfections. Fig. 5 considers the case when $\bar{\xi}_1 = 0$ and $\bar{\xi}_2$ varies. The features which are predominant are that the type one and type two problems yield essentially similar trends and that the parameter ϵ causes a quite uniform shift in the critical load-imperfection curve.

Summary and Conclusions

This paper has presented the initial postbuckling analysis of a plate loaded under the action of two independent in-plane loads. It has been noted that in general terms single mode buckling occurs if the applied loads are of unequal magnitude and multiple mode buckling occurs for equal magnitude loads. One particular aspect of the multiple mode case has been investigated. That is, a two mode case which results from the lowest order coupling in the problem. It has been further noted that for this situation the problem takes the form of the parabolic umbilic catastrophe and thus for a complete representation of the initial postbuckling behavior, a higher-order quartic term in the potential energy as well as independence of the applied loads must be permitted. The influence of these factors are then investigated and it is fair to say that the inclusion or exclusion of the quartic term can alter the postbuckling behavior completely while the variability of the applied loads results in some not unimportant changes in the overall as well as asymptotic character of the critical load-initial imperfection results.

Two additional factors related to this presentation should be mentioned. First, the two mode analysis considered is only part of a more complex multimode situation; however, the coupling terms for the more general case are of fourth order or higher. In addition, the present problem will always exist as a reduction of the more general case. Second, the form of the cubic terms in the potential energy $\alpha\xi_1\xi_2^2$ obtained in the present analysis leads directly to a consideration of the parabolic umbilic. This is not the most general situation as the same results will occur if the cubic terms are of a more complete form $A\xi_1^3 + B\xi_1^2\xi_2 + C\xi_1\xi_2^2 + D\xi_3^3$ where the coefficients A, B, C, D are related such that this cubic form has two equal roots. Thus higher-order terms in the potential energy as well as independence of applied loads may play a role even when the cubic terms in the potential energy are apparently quite complete.

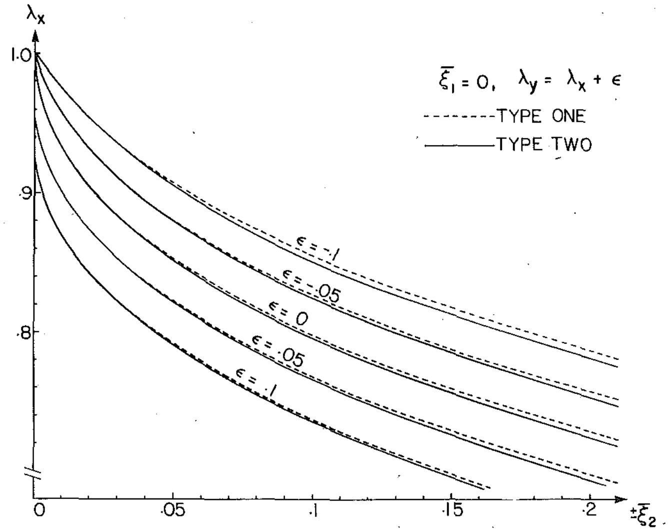

Fig. 5 Critical load-imperfection curves for unequal applied loads ($\lambda_y = \lambda_x + \epsilon$) and $\bar{\xi}_1 = 0$

References

1 Thom, R., *Structural Stability and Morphogenesis,* W. A. Benjamin Inc., 1975.

2 Koiter, W. T., "On the Stability of Elastic Equilibrium," Thesis, Delft, H. J. Paris, Amsterdam, 1945, English translation: NASA TT F-10, 833, 1967, and AFFDL-TR-70-25, 1970.

3 Thompson, J. M. T., and Hunt, G. W., *A General Theory of Elastic Stability,* Wiley, 1973.

4 Budiansky, B., and Hutchinson, J. W., "Dynamic Buckling of Imperfection-Sensitive Structures," *Proceedings of the 11th International Congress of Applied Mechanics,* Munich, Germany, 1964, pp. 636–651; ed., Gortler, H., Springer-Verlag, 1966.

5 Huseyin, K., *Nonlinear Theory of Elastic Stability,* Noordhoff International Publishing, Leyden, 1975.

6 Thompson, J. M. T., and Hunt, G. W., "Toward a Unified Bifurcation Theory," *Journal of Applied Mathematics and Physics, ZAMP,* 1975, pp. 581–604.

7 Sewell, M. J., "Some Mechanical Examples of Catastrophe Theory," *The Institute of Mathematics and Its Applications,* June 1976, pp. 163–172.

8 Huseyin, K., "The Multiple-Parameter Stability Theory and Its Relation

to Catastrophe Theory," *Proceedings of the Interdisciplinary Conference on Problem Analysis in Science and Engineering*, Branin, F., and Huseyin, K., Academic Press, 1977, pp. 229–255.

9 Hansen, J. S., "Some Two-Mode Buckling Problems and Their Relation to Catastrophe Theory," *AIAA Journal*, Vol. 15, No. 11, 1977, pp. 1638–1644.

10 Hui, D., "The Parabolic Umbilic Catastrophe and Its Application in the Theory of Elastic Stability," MASc Thesis, University of Toronto, Jan. 1977.

11 Hui, D., and Hansen, J. S., "The Swallowtail and Butterfly Cuspoids and Their Applications in a Beam on a Nonlinear Foundation," *Quarterly of Applied Mathematics*, to appear.

12 Hilton, P., ed., "Structural Stability, The Theory of Catastrophes, and Applications in the Sciences," *Proceedings of the Conference*, held at Battelle Research Center, 1975, *Lecture Notes in Mathematics*, Vol. 525, Springer-Verlag, 1977.

13 Reissner, E., "On Postbuckling Behavior and Imperfection Sensitivity of Thin Elastic Plates on a Nonlinear Elastic Foundation," *Studies in Applied Mathematics*, Vol. XLIX, No. 1, Mar. 1970, pp. 45–57.

14 Hutchinson, J. W., "Imperfection Sensitivity of Externally Pressurized Spherical Shells," ASME JOURNAL OF APPLIED MECHANICS, Vol. 34, 1967, pp. 49–55.