Mode Interaction of Axially Stiffened Cylindrical Shells: Effects of Stringer Axial Stiffness, Torsional Rigidity, and Eccentricity

This paper deals with the effect of stringer axial stiffness, torsional rigidity, and eccentricity on the panel initial postbuckling behavior and mode interaction of axially stiffened cylindrical shells. As far as the local panel mode is concerned, the cylinder is taken to be integrally stiffened and the postbuckling redistribution of the axial load-carrying capacity between the skin and stringers is investigated. It is found that although the buckling analysis is more or less identical to previous analyses, there are significant changes in the quartic term of the potential energy of the local mode and hence, the imperfection-sensitivity of the structure is altered. These changes are due to the interaction between the stringers and the skin in the postbuckling analysis. In addition, to assess the effect of a nonlinear prebuckling state resulting from the presence of local imperfections, Koiter’s theory of amplitude modulation of the local mode is applied to an example problem of interest.

Introduction

It has been theoretically observed that the interaction between simultaneous buckling modes may result in highly imperfection-sensitive structures with the conclusion that coincident buckling between competing modes may not yield an optimal design. Of particular interest here are investigations of the imperfection-sensitivity of periodically stiffened structures such as stringer or ring-reinforced cylindrical shells or stringer-reinforced flat plates. The reason for this interest is that local buckling of the panel between stringers and overall buckling of the structure may occur more or less concurrently.

Using the general theory of elastic stability first introduced by Koiter [1-3], several authors have studied a variety of mode interaction problems and imperfection-sensitivity of optimal structural designs [4-14]. Of these, one of the most commonly investigated structural optimization problems of significant practical interest is that of an integrally stringer reinforced cylindrical shell subjected to axial compression. For this problem, the imperfection-sensitivity of overall buckling has been examined [15, 16] using a smeared-out analysis which is valid when the number of circumferential half waves involves at least two or more stringers. This last assumption can be verified a posteriori and is generally accepted that many practical structures satisfy this requirement. However, in the local panel analysis, the structure must be treated as integrally stiffened; thus necessitating assumptions concerning the behavior of the stringers. For example, in Koiter’s study [17] of the local buckling and postbuckling behavior of stiffened panels, it was assumed that the stringers cannot bend out-of-plane and the tangential bending stiffness of the stringer was neglected. In addition, the applied load was considered to act only on the skin and not on the stringers and thus slippage between stringer and skin in the axial direction was permitted, although, as it turns out, this assumption is quite reasonable when analyzing the local sheet mode alone. Later, Stephens [18] incorporated the possibility that the stringers provide rotational constraint of the skin about the line of attachment.

In the present work, Stephens’ model has been modified such that stringers and skin are now subjected to the same applied stress and postbuckling redistribution of the axial load due to the interaction between skin and stringers is taken into account in the local panel mode analysis. This modification is essential in the analysis of the interaction between overall and local buckling because the axial constraint between stiffeners and sheet has a significant stabilizing effect on the local sheet mode in the post-buckling range. The results of Koiter’s analysis in [14] and the curves associated with Koiter’s and Stephens boundary conditions will be shown (Fig. 3) to yield unduly conservative buckling loads.

As far as the overall mode post-buckling analysis is concerned, it is identical to that presented in [15] and requires no modification. The final portion of the analysis studies the two-mode interaction problem...
using the quartic terms found in the foregoing local and overall mode analyses in conjunction with Koiter’s theory of amplitude modulation of the local model [14]. Unlike Koiter’s general theory [1], the theory of local amplitude modulation is valid for larger values of the imperfection amplitude. It should be noted that local distortion of the stringer cross section has been ignored in the analysis.

Potential Energy of the Cylindrical Panel

Consider an axially stiffened circular cylindrical shell subjected to combined axial compressive and lateral pressure loads. The prebuckling state for the local mode is assumed to be linear so that the cylindrical panel is in a state of uniform compression prior to buckling. The potential energy of the skin based on the Donnell assumptions and the nonlinear strain-displacement relations can be written as [1]:

\[ PE = (E t^2)\left( P_y[u] + P_y[u] + P_y[u] + P_y[u] + \ldots \right) \]

where

\[ P_y[u] = \left( \frac{1}{2(1 - \nu)^2} \right) \int_0^\theta \int_0^R [u_{x x}^2 + (v_y + w)^2 + 2v(u_x + w) + \chi (1 - \nu)(u_y + w)^2 + \nu u_{x x} + \nu w_y + 2v(u_x + w)] dx dy \]

or

\[ P_y[u] = \left( \frac{1}{2(1 - \nu)^2} \right) \int_0^\theta \int_0^R \left( c_1 + c_2 \right) w_{y y}^2 dx dy \]

\[ P_y[u] = \left( \frac{1}{1 - \nu^2} \right) \int_0^\theta \int_0^R \left[ c w_{y y}^2 + 2v^2 e (e_y)^2 + \nu u_{x x} + \nu w_y + (u_x + w) \right] dx dy \]

\[ P_2[u] = \left[ 1/(2q_o) \right] \int_0^\theta \int_0^R \left( [E_n a_s/\alpha_s] \right) [u_x = 2e(e_y)w_{y x}]^2 + \beta_s (d_s/R) (w_{x x})^2 + \nu (q_o) (w_{x y})^2 \] dx dy

\[ P_3[u] = \left[ 1/(q_o) E_n a_s (d_s/R) \right] \int_0^\theta \int_0^R \left[ c w_{y y}^2 + 2v^2 e (e_y)^2 \right] (t/R) (w_{x x})^2 \] dx dy

\[ Z = \left[ L^2/(R\varepsilon) \right] \sqrt{1 - \nu^2}, \text{ Batdorf parameter} \]

\[ \alpha_s = A_s/(d_s^2), \text{ area ratio} \]

\[ \beta_s = \text{ angle in radians between adjacent stringers} \]

\[ \beta_s = (E_s I_s)/(d_s), \text{ out-of-plane bending stiffness ratio} \]

\[ \gamma_s = (q_o G_o P_s)/(DR), \text{ tangential bending stiffness ratio} \]

\[ \theta = q_o M_s/(D t), \text{ flatness parameter} \]

\[ \lambda = \text{ ratio of the wavelength of the local mode in the circumferential direction to that in the axial direction} \]

\[ \lambda* = \text{ as defined in the text} \]

\[ \nu = \text{ Poisson’s ratio} \]

\[ \xi_s = \text{ amplitude of the overall mode with respect to the skin thickness} \]

\[ \sigma = \text{ (Rt)/(EI)}, \text{ non-dimensional axial stress relative to the lighter weight unstiffened cylindrical shell} \]

\[ \tau = \text{ torsional factor which varies from 0.14 for square-shaped stringer to 0.353 for thin-walled rectangular stringer} \]

Note: A comma preceding subscripts indicates differentiation with respect to the variables shown.
P_0[u] = \left[ \frac{1}{2} (2a + \sigma) E \alpha d_0 (\delta t) \right] \int_0^{2 \pi} |c^2(w, x)|^4 \, dx + 2 \pi (2a + \sigma) E \alpha d_0 (\delta t) |c^2(w, x)|^4 + 4 \pi (2a + \sigma) E \alpha d_0 (\delta t) |c^2(w, x)|^3 \, dx \tag{3}

(Cont.)

It should be noted that not all the aforementioned stringer energy contribution is considered by both Koiter and Stephens and it will be shown that, in general, it is highly significant in the computation of the quartic term of the potential energy of the local mode. Consistent with Koiter's general theory of elastic stability, the total displacements can be expressed as the sum of the prebuckling deformation, the buckling mode, and the second-order field in the following form:

\[ u(x, y) = u_0(x, y) + (\delta t) [u_1(x, y) + (\delta t)^2 u_2(x, y)] \tag{4} \]

where \( \delta t \) is the amplitude of the buckling mode normalized with respect to the skin thickness. In the postbuckling regime, it was shown [1, 2] that the equilibrium states are specified by

\[ \left( 1 - \frac{a}{\sigma} \right) (\delta t) + a (\delta t)^2 + \frac{3}{5} (\delta t)^3 = (a/\sigma \varepsilon) (\delta t) \tag{5} \]

where the \( a \) and \( b \) coefficients are related to the cubic and quartic terms of the potential energy, respectively, \( \delta \) is the amplitude of the local imperfection which is taken to be of the same form as the buckling mode. In this expression, the cubic term \( a \) of the local mode vanishes, provided that the number of axial half waves is even, the number of stringers is even or the shell is infinitely long. Since local buckling of a stiffened shell of sufficiently high \( Z > 200 \) generally results in many local buckles, and due to symmetry considerations, the cubic term is assumed to be negligible. Thus a negative value of \( b \) indicates that the structure is imperfection-sensitive while a positive value indicates a lack of imperfection-sensitivity. In order to compute the \( b \) coefficient, it is necessary to determine the second-order field in addition to the buckling mode. Furthermore, the second-order field \( (u_2, v_2, w_2) \) is obtained by imposing the appropriate orthogonal conditions and by minimizing the quadratic and cubic terms of the potential energy. That is, the minimization

\[ \delta P_0[u_2] + \delta P_0[u_3] = 0 \]

yields the following three differential equations [1, 19],

\[ w_{xx} + \frac{1}{2} (1 - \nu) w_{yy} + \frac{1}{2} (1 + \nu) w_{xy} + w_{xx} = -c^2 \omega_0 w_{xx} + (1 + \nu) w_{xy} w_{x} + (1 - \nu) w_{xy} w_{y}, \]

\[ \frac{1}{2} (1 + \nu) w_{xx} + w_{yy} + \frac{1}{2} (1 - \nu) w_{xy} + w_{yy} = -c^2 \omega_0 w_{xy}, \]

\[ (u_{xy}) \left( \frac{1}{1 - \nu^2} \right) + (w_{xy}) \left( \frac{1}{1 - \nu^2} \right) + \nabla^2 w_{xx} + 2 S w_{xx} \]

\[ -2p w_{xx} + (w_{xy}) \left( \frac{1}{1 - \nu^2} \right) = 2c \left( f_{xy} w_{xx} + f_{xy} w_{xy} \right) + \left( \nu w_{xx} + \frac{w_{xy}^2}{2 (1 - \nu^2)} \right) \tag{6} \]

where \( S \) is defined to be

\[ S = (\sigma) \left( \frac{1 + \alpha_s}{1 + E \alpha_s} \right) - (\sigma_0) \left( \frac{E \alpha_s}{1 + E \alpha_s} \right) \tag{7} \]

and it should be noted that \( S = \sigma \) in the important special case \( p = 0, E_0 = 1, \alpha_s = 0 \) or in the case \( \alpha_s = 0 \). Furthermore, the Airy stress function \( f_{xy} \) is related to \( u_{xy} \) and \( w_{xy} \) by

\[ (1 - \nu^2) (f_{xx}) = u_{xx} + (u_{xy} w_{xx} + w_{xy}), \]

\[ (1 - \nu^2) (f_{yy}) = u_{yy} + (u_{xy} w_{xy} + w_{xy}), \]

\[ -2 (1 + \nu) (f_{xy}) = u_{xy} + \nu w_{xy} \tag{8} \]

In addition to the continuity requirements of the displacements at the stringer edge, i.e., \( u^+ = u^- \), \( v^+ = v^- \), \( w^+ = w^- \), and \( w^+_y = w^-_y \), the four jump conditions at the stringer edge (presented in dimensional form for clarity purposes) are given by

\[ U = 0 \text{ or } N^+_s - N^-_s = (E_0 A_s) \left( u^+_s - u^-_s \right), \]

\[ V = 0 \text{ or } N^+_s - N^-_s = E_0 I_s V_{xxxx} = 0, \]

\[ W = 0 \text{ or } (D W_{xyy} - W_{yy}) + E_0 N_{xxx}, \]

\[ W_y = 0 \text{ or } D (W_{yy} - W_{yy}) + E_0 N_{xxx} + \epsilon^2 N_s W_{xxx} + \epsilon^2 N_s W_{yy} = 0 \tag{9a} \]

Where subscripts "+" and "−" denote the positive and negative sides of a stringer as measured in the circumferential direction. Moreover, \( N_s \) is the axial force applied at the centroid of the stringer and it is given by

\[ N_s = (E_0 A_s) \left( \epsilon_s - \epsilon_s W_{xx} + \frac{1}{2} \epsilon^2 W_{xy} \right) \tag{9b} \]

The desired jump conditions for the buckling state and for the second-order field computations are obtained by substituting \( u = u_0 + (\delta t) u_1(x, y) + (\delta t)^2 u_2(x, y) \) into the foregoing conditions and collecting terms involving \( (\delta t) \) and \( (\delta t)^2 \), respectively. They will be presented subsequently.

### Classical Local Buckling Load

As the applied load is increased from zero, the classical buckling load is defined to be the smallest eigenvalue which causes the quadratic terms of the potential energy to vanish. Due to symmetry considerations, the local buckling mode \( u_0(x, y) \) must be composed of either even or odd functions. In the spherical shell, \( (x, y) \) is an even function with respect to the stringer edge. This implies that \( u_0(x, y) = 0 \) and thus the corresponding jump conditions can be neglected. If the shell is assumed to be sufficiently long such that there are many local waves in the axial direction, then the buckling mode is expressible in the form

\[ u_c(x, y) = u_c(y) \cos \left( \frac{\lambda x}{20} \right), \]

\[ v_c(x, y) = v_c(y) \sin \left( \frac{\lambda x}{20} \right), \]

\[ w_c(y) = w_c(y) \sin \left( \frac{\lambda x}{20} \right), \] \tag{10}

\[ \text{The boundary conditions at the stringer edge are} \]

\[ u_c(y = 0) = 0 \]

\[ w_c(y = 0) = 0 \]

\[ v_c(y = 0) = 0 \]

\[ \beta \left[ \frac{\lambda x}{20} \right] \text{where } \beta \text{ is the nondimensional stringer tangential bending stiffness ratio, and } \gamma_s \text{, is related to the torsional rigidity of the stringer. Since the tangential bending energy of the stringer is usually small compared with the out-of-plane bending energy and the purpose of the present paper is to study the effect of stringer eccentricity and cross-sectional area on the imperfection-sensitivity of the cylindrical panel, the quantity } \beta \frac{\lambda x}{20} \text{ is set to zero for comparison purposes with Stephens' paper [18]. Making such a simplification, the present eigenvalue problem is identical to that examined by Stephens. In the computation of the buckling load, the mixed formulation is used instead of the UVW formulation due to its simplicity. The ordinary differential equations are}

\[ w_{c}(y) \left( \frac{\lambda x}{20} \right) - \pi \left( 1 - \nu^2 \right) \left( f_{xy} \right) \left( \frac{\lambda x}{20} \right) \text{subject to the boundary conditions at } } \]

\[ \text{The ordinary differential equations are} \]

\[ w_{c}(y) \left( \frac{\lambda x}{20} \right) = 0 \]

\[ v_{c}(y) \left( \frac{\lambda x}{20} \right) = 0 \]

\[ u_{c}(y) \left( \frac{\lambda x}{20} \right) = 0 \]

\[ \frac{\lambda x}{20} \text{subject to the boundary conditions at } } \]

\[ w_{c}(y) \left( \frac{\lambda x}{20} \right) = 0 \]

\[ v_{c}(y) \left( \frac{\lambda x}{20} \right) = 0 \]

\[ u_{c}(y) \left( \frac{\lambda x}{20} \right) = 0 \]

\[ \frac{\lambda x}{20} \text{subject to the boundary conditions at } } \]
where \( z = [\lambda/(2\theta)]^2 \) and \( S \) is defined in equation (7). The four boundary conditions at the stringer edge are

\[
\begin{align*}
  w_x(y = 0) &= 0, \quad (z/2)g_M w_c(y = 0) - w_c(y = 0)_{,yy} = 0, \\
  f_c(y = 0) &= 0, \quad f_c(y = 0)_{,yy} = 0
\end{align*}
\]

while the boundary conditions at the midpanel are

\[
\begin{align*}
  w_x(y = \pi\theta) &= 0, \quad w_x(y = \pi\theta)_{,yy} = 0, \\
  f_c(y = \pi\theta) &= 0, \quad f_c(y = \pi\theta)_{,yy} = 0
\end{align*}
\]

The foregoing ordinary differential equations with boundary conditions can be solved using a central finite-difference scheme in conjunction with the computation of the determinants as was done by Stephens and others using a modification of Potter's method [21]. However, since only the smallest eigenvalue is of interest, it is felt that the previous method can be replaced in the following way by a more efficient technique. First, the resulting system of linear finite-difference equations were rearranged into the standard form \( Ax + Bx = 0 \) where \( A \) and \( B \) are both banded matrices and \( D \) is singular (the matrix \( B \) should not be confused with the interaction parameter \( D \) employed in [14]). The shifted inverse power method was then used to compute the eigenvector and eigenvalue, details of which can be found in [22, 23]. This obviates the need to compute determinants and drastically reduces the computation time. Second, a more efficient linear equation solver [20] was used instead of Potter’s method. The classical buckling load was found by computing the smallest eigenvalue for all possible wavelengths \( \lambda \) (usually the optimum \( \lambda \) ranges between one and two). It was observed that the quartic term measured by the interaction parameter \( B \) is both larger than was the buckling load. Consequently, 61 finite-difference equations were rearranged into the standard form \( oBx = 0 \) and hence the corresponding jump conditions can be neglected.

\[
\begin{align*}
  w_{,yy}(y = 0) &= 0, \quad w_{,yy}(y = 0) = 0,
\end{align*}
\]

Again, using symmetry at the stringer edge, \( u_p(y = 0) = 0 \) and \( u_p(y = 0)_{,y} = 0 \) and the corresponding jump conditions with respect to \( u_2 \) and \( u_3 \), may be neglected. The jump conditions with respect to \( u_2 \) and \( u_3 \) are

\[
\begin{align*}
  w_{,yy}(y = 0) &= 0, \quad w_{,yy}(y = 0) = 0
\end{align*}
\]

Note that for the constants \( u_0 \) and \( w_0 \) [equation (15)] cannot be obtained from the jump conditions but are obtained by minimizing

\[
\begin{align*}
  (P_2[u_2] + P_2[u_3] + P_2[u_2])_{,yy} + (P_2[u_2] + P_2[u_3])_{,yy}
\end{align*}
\]

with respect to \( u_0 \) and \( w_0 \). After some algebra, they are found as [22],

\[
\begin{align*}
  u_0 &= (R_1 - R_2)/[(1 - \nu^2) (1 + E_m c)] \\
  w_0 &= [(1 + (\nu^2)(E_m c))(R_2 - \nu R)/[(1 - \nu^2) (1 + E_m c)]
\end{align*}
\]

(21a)

where \( R_1 \) and \( R_2 \) are defined by

\[
\begin{align*}
  R_1 &= -c/2 \left[ x H_1 + \nu H_2 - [(1 - \nu^2) (E_m c)] \right] \\
  R_2 &= -c/2 \left[ x H_1 + H_2 \right] \\
  H_1 &= [1/(\theta^2)] \int_0^{\theta} \left[ w(x,y) \right]^2 dx, \\
  H_2 &= [1/(\theta^2)] \int_0^{\theta} w(x,y) \right]^2 dx
\end{align*}
\]

(21b)

In the special case of zero torsional rigidity, one obtains,

\[
\begin{align*}
  H_1 &= 1/2, \quad H_2 = 1/(\theta^2), \quad w(x, y = 0) = 1/(2\theta), \quad z = [\lambda/(2\theta)]^2
\end{align*}
\]

(21c)

so that

Now, for the stringer edge conditions, \( u_p(y = 0) = 0, w_p(y = 0) = 0, w_p(y = 0)_{,y} = 0 \) and \( w_p(y = 0)_{,yy} = 0 \). The uncoupled alpha and beta problems are then solved using a central finite-difference scheme.

\[
\begin{align*}
  u_p(y)_{,yy} + [c/((1 + \nu) c)] (\lambda/\theta) u_p(y)_{,yy} + [(\lambda/\theta)^4 - (2\lambda/\theta)^2] u_p(y)_{,yy}
\end{align*}
\]

(18)

\[
\begin{align*}
  w_p(y)_{,yy} + [c/(1 + \nu)] (\lambda/\theta) w_p(y)_{,yy} + |\beta_2 \pi (\lambda/\theta)^4 + (E_m c) c (c) (2\pi R/R) |
\end{align*}
\]

\[
\begin{align*}
  \times (\lambda/\theta)^2 |
\end{align*}
\]

(19)
\[ u_0 = \frac{1}{(1 + E_\alpha \alpha_s)} [-c/(16 \theta^2)] [1 + (E_\alpha \alpha_s)(e_\alpha/t)^2(t/R)/c(16 \theta^2)] \]
\[ w_0 = \frac{1}{(1 + E_\alpha \alpha_s)} [-c/(16 \theta^2)] [1 + (E_\alpha \alpha_s) - (c/\theta^2)(e_\alpha/t)^2(t/R)] \]

(22)

It can be seen that the constants \( u_0 \) and \( w_0 \) converge to Koiter's result [17] for no applied pressure, \( \lambda = 1, e_\alpha/t = 0, \) and \( \alpha_s = 0. \)

**Initial Postbuckling Behavior**

The potential energy expression of a single mode symmetric system, evaluated at the classical buckling load (note that the initial imperfection is taken to be of the same shape as the buckling mode)

\[ \frac{P.E.}{(E \rho^2)} = (A_4)(\delta/t)^4 + (\sigma - \sigma_c)(\delta/t)^3 + (2\sigma_c)(\delta/t)^2 b/(\delta/t) \]

(23a)

where \( b/(\delta/t) \) is the amplitude of the imperfection. Moreover, in the present stringer reinforced cylindrical panel problem, \( A_4 \) and \( d \) are computed from

\[ A_4 = (P_s^0[u_c] - P_s^0[u_2] + P_s^0[u_2]) \]
\[ + (P_s^0[u_c] - P_s^0[u_2] + P_s^0[u_2]) \]
\[ d = \frac{\delta}{\sigma_c} P_s^0[u_c] + \frac{\delta}{\sigma_c} P_s^0[u_2] \]

(23b)

Thus the equilibrium equation can be written as,

\[ b/(\delta/t)^3 + [1 - (\sigma / \sigma_c)](\delta/t) = (\sigma / \sigma_c)b/(\delta/t) \]

(24a)

where the \( b \) coefficient of the present local problem is

\[ b = (4A_4)/[(-2d)(\delta/t)] \]

(24b)

The structure is imperfection-sensitive if the \( b \) coefficient is negative. Since the buckling mode \( u_c \) and the second-order field \( u_2 \) are now known quantities, it is straightforward to compute the individual terms of the \( b \) coefficient. The quartic term of the potential energy for the skin (from stringer to midpanel)

\[ Q_1 = P_s^0[u_c] - (P_s^0[u_2] + P_s^0[u_2]) \]
\[ \times \int_0^{\delta^*} \left\{ (3A^4)/(16 \theta^4) \right\} w_c(y)^4 + 3w_c(y)^2 \]
\[ + \left\{ (3A^4)/(16 \theta^4) \right\} w_s(y)^4 \] (25a)

\[ Q_2 = \left\{ \frac{1}{(2(1 - \rho^2))} \right\} \frac{L}{2d} \int_0^{\delta^*} \left\{ w_s(y)^2 \right\} + w_s(y)^2 \] (25b)

In addition, the stringer contribution to the quartic term is (upon dividing the result by two since only half a cylindrical panel between adjacent stringers has been considered)

\[ \{P_s^0[u_c] - (P_s^0[u_2] + P_s^0[u_2]) \} \times \frac{1}{(1 + E_\alpha \alpha_s)} \]
\[ \times \left\{ \frac{1}{(1 + E_\alpha \alpha_s)} \right\} \]

(26)

Also for the skin,

\[ P_s^0[u_c] = (\sigma_c) \]
\[ \times \int_0^{\delta^*} w_c(y)^2 dy \]

(27a)

and for the stringer, again, upon dividing by two,

\[ P_s^0[u_c] = (\sigma_c) \]
\[ \times \int_0^{\delta^*} w_c(y)^2 dy \]

(27b)

**Mode Interaction by Amplitude Modulation of the Local Mode**

In order to investigate imperfection-sensitivity due to local and overall mode interactions, Koiter's theory of amplitude modulation of the local mode [14] was used to compute the two-mode potential energy expression. This necessitates incorporating the appropriate terms to account for axial stiffness, torsional rigidity, and eccentricity of the stringers. For brevity, only the final form of the potential energy expression is given (for details, refer to [22]),

\[ \frac{P.E.}{(E \rho^2)} = \left\{ [1 - (\lambda^* / \rho)](a_i^2) + (1 - \lambda^*) \right\} \int h(x, y) dy \]

(28)

Furthermore, \( a_1, a_0, B, C, \lambda^*, \) and \( r \) are defined to be

\[ \lambda^* = (\xi_1 \sqrt{Z_0} / Z_0) \]

(29a)

\[ W_{(x)} = (t) (x, y) \sin (m \pi X(L))/w_c(y) \]

(29b)

where \( dS = (dX dY)/(2 \pi RL) \), \( h(x, y) \) is the modulated amplitude of the local mode normalized with respect to the skin thickness, \( b_2 \) is the \( b \) coefficient of the local mode. The overall mode [15] and the modulated local mode are

\[ W_{(x)} = (t) (x, y) \sin (m \pi X(L))/w_c(y) \]

(30)

where \( \alpha_i \) and \( \alpha_0 \) are the overall and local buckling loads, respectively, \( b_2 \) the \( b \) coefficient of the overall mode. In the foregoing expressions, \( Z_1, Z_2, \) and \( Z_3 \) are defined as

\[ Z_1 = [\alpha_i(8c)](1 + \alpha_c(m \pi)^2) \]

(31)

where \( H_1 \) and \( H_2 \) are defined in equations \((21b) \) and \((21c) \) and \( S_1, S_2 \) are (using classical simple support conditions at \( X = 0, L) \)

\[ S_1 = \left\{ [A_{nx}(m \pi)^2(t/R)^2] - A_{nx}^2 t/(2 \pi RL)^2 \right\} \]

(32)

\[ S_2 = \left\{ [A_{ny}(m \pi)^2(t/R)^2] - A_{ny}^2 t/(2 \pi RL)^2 \right\} \]

\[ + \left\{ [A_{ny}(m \pi)^2(t/R)^2] - A_{ny}^2 t/(2 \pi RL)^2 \right\} \]
The torsional rigidity ratio can be obtained from
\[
(1 - \lambda^*_b) = \frac{q_0}{M_b} = \frac{(1 - \lambda^*_b)(1 - \lambda^*_b)}{(2B - 3b)(1 - \lambda^*_b)}
\]

Note that since the stringers are constrained to be of rectangular shape
published literature \([6, 9, 14, 24, 25]\) details of which can be found in

\[
\approx \frac{B - 3b(1 - \lambda^*_b)}{2B - 3b(1 - \lambda^*_b)}
\]

In the event that the \(b\) coefficient of the local panel mode turns out
be positive, the previous relation yields a lower bound given by

\[
L.B. = (r)[1 - [(2B)/(3b)]]
\]

**Example Problem**

To illustrate the effects of stringer eccentricity, axial stiffness, and
torsional rigidity on buckling, imperfection-sensitivity, and mode
interaction, example problems have been chosen from the relevant
published literature \([6, 9, 14, 24, 25]\) details of which can be found in
[22]. However, for brevity, only one example will be presented here
based on the following shell parameters taken from Byskov and
Hutchinson \([6]\). The fixed parameters are

\[
\alpha_a = 0.7, \quad Z = 810.846, \quad R/t = 860, \quad v = 0.3, \quad q_0 = 52.998
\]

\[
R/L = 1.0, \quad E_a = 1.0, \quad t_{s}/t = 4.09, \quad \beta_t = 10.6658.
\]

Note that since the stringers are constrained to be of rectangular shape
and \(a, t_{s}/t\) are held constant, then changing the number of stringers
requires changes in the stringer height \(Q_a\) as well as

\[
\theta = q_0/M_a, \quad Q_a/t = [\alpha_a/(t_{s}/t)](2\pi/M_a)(R/t),
\]

\[
e_{s} = (1/2) [1 + (Q_a/t)]
\]

\[
\beta_s = (E_a\alpha_a)(c^2/3)(Q_a/t), \quad d_{s}/t = (t_{s}/t)(Q_a/t)(1/\alpha_a)
\]

The torsional rigidity ratio can be obtained from

\[
\gamma_s = \frac{[12\alpha_a\theta]}{[(d_{s}/t)(c^2/3)]} \text{ minimum } (\beta_s, \beta_t)/(1 + \nu)
\]

\[
= (E_a\alpha_a)(c^2/3)(8\pi c)(c^2/4)
\]

where \(\alpha_s\) is the torsional constant which ranges from 0.14 for square

shape stringers to 0.333 for thin-walled stringers. The second term
in equation (37) refers to the axial shortening of the stringer due to
twisting and is usually quite small compared with the first term.

Using these values, the classical buckling load normalized with
respect to the lighter weight unstiffened cylindrical shell classical
buckling load is plotted against the flatness parameter \(\theta\) and number
of stringers \(M_b\) in Fig. 1. It should also be pointed out that this analysis
differs from that of Stephens since the permissible wave number \(\lambda\)
is treated as discrete and axial shortening of the stringers due to
twisting is included. However, buckling loads close to Stephens' result
were obtained, thus substantiating that significant increases in local
panel buckling loads arise primarily from the inclusion of stringer
torsional rigidity. Based on the smeared-out overall buckling analysis
of \([15]\) (classical simple support, \(X = 0, L\)), it can be seen in Fig. 1 that
the simultaneous occurrence of local and overall buckling occurs at
\(\theta = 0.78\) (\(M_b = 68\)) with \(\sigma = 1.66\) instead of \(\theta = 0.64\) (\(M_b = 83\)) with
\(\sigma = 1.43\) found in \([6, 24]\).

When the variation in the local \(b\) coefficient with \(\theta\) (Fig. 2) is con­sidered, the effect of stringer eccentricity and axial stiffness shows
essentially a constant shift from Stephens' result \([18]\). This is due to
the fact that the area ratio \(\alpha_a\) is kept constant and that the percentage
transfer of the axial stress from the skin to the stringer is a function
of \(\alpha_s\). By examining the change in the axial strain via the constant \(u_0\)
in equation (15), an approximate formula for the aforementioned
percentage transfer of the axial stress is \([22]\),

\[
\text{percent transfer} = [\alpha_s/(1 + \alpha_s)] (100\%)
\]

Moreover, the transition between stable and unstable postbuckling
behavior is indicated by the sign change in the \(b\) coefficient which
occurs at \(\theta = 0.89\) (with \(M_b = 80\)) rather than at \(\theta = 0.775\) (with \(M_b\)
= 68) which is predicted by Stephens \([18]\). From Fig. 1, it may be seen
that coincident buckling occurs at \(\theta = 0.78\) which corresponds to
\(\beta = 1.43\) found in \([6, 24]\).
Furthermore, it should be noted that the value of the lower bound estimates (from equation (34)) when \( \theta \) is positive.

off" quite rapidly for imperfection amplitudes likely to be encountered torsional rigidity of the stringers. The present curves tend to "level off" and the ratio of the overall to the local buckling load \( r \). Based on the computation of these parameters for the three designs being considered \( (M_b = 64, 68, \text{and } 72) \), the bifurcation load normalized with respect to the skin thickness versus local imperfection is also considered \( (\gamma_{\text{local}} > 0) \), it is evident that very large changes in the \( b \) coefficient occur.

At this point, it is of particular interest to study the effect of a nonlinear prebuckling state (due to the presence of a local imperfection alone) on the previous local and overall mode interaction problem. The four parameters required to specify the two-mode potential energy expression are: the transformed \( b \) coefficient of the overall mode \( C_1 \), the \( b \) coefficient of the local mode \( b_2 \), the interaction cubic term \( B \), and the ratio of the overall to the local buckling load \( r \). Based on the computation of these parameters for the three designs being considered \( (M_b = 64, 68, \text{and } 72) \), the bifurcation load normalized with respect to the classical buckling load of the local mode \( \sigma_{\text{local}} \) (it should be cautioned that \( \sigma_{\text{local}} \) differs between the present and Koiter's 1956 analyses) is plotted against the amplitude of the local imperfection in Fig. 3. Comparing these solutions, one can conclude that significant differences arise due to the inclusion of axial stiffness and torsional rigidity of the stringers. The present curves tend to "level off" quite rapidly for imperfection amplitudes likely to be encountered in practice. Table 1 summarizes the essential results together with the lower bound estimates (from equation (34)) when \( b_2 \) is positive. Furthermore, it should be noted that the value of the \( b \) coefficient of the overall mode has no effect whatsoever in Fig. 3 (see [14]). It was also found that each circumferential half wave of the overall mode involved about two to three stringers, thus indicating that the smeared-out overall mode analysis is a reasonable approximation for this case.

Finally, it should be emphasized that several other example problems based on different shell configurations have been examined. It was found that all these investigations yield qualitatively similar results. Experimental results confirming the present buckling and initial postbuckling behavior of the local mode can be found in [25].

**Acknowledgments**

The authors wish to gratefully acknowledge the contributions made at various stages in the development of this work by Prof. J. W. Hutchinson of Harvard University, who first suggested the problem to us. In addition, Prof. W. T. Koiter provided valuable guidance at the outset of the program. Financial support of our research was received from the National Sciences and Engineering Research Council under Grants A-2783 and A-3663.

**References**


