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Mode Interaction of Axially Stiffened Cylindrical Shells: Effects of Stringer Axial Stiffness, Torsional Rigidity, and Eccentricity

This paper deals with the effect of stringer axial stiffness, torsional rigidity, and eccentricity on the panel initial postbuckling behavior and mode interaction of axially stiffened cylindrical shells. As far as the local panel mode is concerned, the cylinder is taken to be integrally stiffened and the postbuckling redistribution of the axial load-carrying capacity between the skin and stringers is investigated. It is found that although the buckling analysis is more or less identical to previous analyses, there are significant changes in the quartic term of the potential energy of the local mode and hence, the imperfection-sensitivity of the structure is altered. These changes are due to the interaction between the stringers and the skin in the postbuckling analysis. In addition, to assess the effect of a nonlinear prebuckling state resulting from the presence of local imperfections, Koiter's theory of amplitude modulation of the local mode is applied to an example problem of interest.

Introduction

It has been theoretically observed that the interaction between simultaneous buckling modes may result in highly imperfection-sensitive structures with the conclusion that coincident buckling between competing modes may not yield an optimal design. Of particular interest here are investigations of the imperfection-sensitivity of periodically stiffened structures such as stringer or ring-reinforced cylindrical shells or stringer-reinforced flat plates. The reason for this interest is that local buckling of the panel between stringers and overall buckling of the structure may occur more or less concurrently.

Using the general theory of elastic stability first introduced by Koiter [1-3], several authors have studied a variety of mode interaction problems and imperfection-sensitivity of optimal structural designs [4-14]. Of these, one of the most commonly investigated structural optimization problems of significant practical interest is that of an integrally stringer reinforced cylindrical shell subjected to axial compression. For this problem, the imperfection-sensitivity of overall buckling has been examined [15, 16] using a smeared-out analysis which is valid when the number of circumferential half waves involves at least two or more stringers. This last assumption can be verified *a posteriori* and it is generally accepted that many practical structures satisfy this requirement. However, in the local panel analysis, the structure must be treated as integrally stiffened; thus necessitating assumptions concerning the behavior of the stringers. For example, in Koiter's study [17] of the local buckling and post-buckling behavior of stiffened panels, it was assumed that the stringers cannot bend out-of-plane and the tangential bending stiffness of the stringer was neglected. In addition, the applied load was considered to act only on the skin and not on the stringers and thus slippage between stringer and skin in the axial direction was permitted, although, as it turns out, this assumption is quite reasonable when analyzing the local sheet mode alone. Later, Stephens [18] incorporated the possibility that the stringers provide rotational constraint of the skin about the line of attachment.

In the present work, Stephens' model has been modified such that stringers and skin are now subjected to the same applied stress and postbuckling redistribution of the axial load due to the interaction between skin and stringers is taken into account in the local panel mode analysis. This modification is essential in the analysis of the interaction between overall and local buckling because the axial constraint between stiffeners and sheet has a significant stabilizing effect on the local sheet mode in the post-buckling range. The results of Koiter's analysis in [14] and the curves associated with Koiter's and Stephens boundary conditions will be shown (Fig. 3) to yield unduly conservative buckling loads.

As far as the overall mode post-buckling analysis is concerned, it is identical to that presented in [15] and requires no modification. The final portion of the analysis studies the two-mode interaction problem

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using the quartic terms found in the foregoing local and overall mode analyses in conjunction with Koiter's theory of amplitude modulation of the local model [14]. Unlike Koiter's general theory [1], the theory of local amplitude modulation is valid for larger values of the imperfection amplitude. It should be noted that local distortion of the stringer cross section has been ignored in the analysis.

Potential Energy of the Cylindrical Panel

Consider an axially stiffened circular cylindrical shell subjected to combined axial compressive and lateral pressure loads. The prebuckling state for the local mode is assumed to be linear so that the cylindrical panel is in a state of uniform compression prior to buckling. The potential energy of the skin based on the Donnell assumptions and the nonlinear strain-displacement relations can be written as [1]:

$$PE = (Et^3)\,(P_2{}^0[u] + P_2{}'[u] + P_3{}^0[u] + P_4{}^0[u] + \ldots) \tag{1a}$$

where

$$\begin{aligned}
P_2{}^0[u] &= \left(\frac{1}{2(1-\nu^2)q_0{}^2}\right) \\
&\times \int_0^{\pi\theta}\int_0^{q_0L/R} \{u_{,x}{}^2 + (v_{,y}+w)^2 + 2\nu(u_{,x})\,(v_{,y}+w) \\
&+ \tfrac{1}{2}(1-\nu)\,(u_{,y}+v_{,x})^2 + (c^2/3)\,[w_{,xx}{}^2 + w_{,yy}{}^2 \\
&+ 2\nu\, w_{,xx}w_{,yy} + 2(1-\nu)w_{,xy}{}^2]\}dx\,dy \\
P_2{}'[u] &= \left(\frac{1}{2(1-\nu^2)}\right)\int_0^{\pi\theta}\int_0^{q_0L/R}\{(c_1+\nu c_2)\,(w_{,x}{}^2) \\
&+ (c_2+\nu c_1)\,(w_{,y}{}^2)\}dx\,dy \\
P_3{}^0[u] &= \frac{c}{(1-\nu^2)\,(q_0)^2}\int_0^{\pi\theta}\int_0^{q_0L/R}\{(w_{,x}{}^2)\,[u_{,x}+\nu(v_{,y}+w)] \\
&+ (w_{,y}{}^2)\,[\nu u_{,x} + (v_{,y}+w)] \\
&+ (1-\nu)\,(w_{,x}w_{,y})\,(u_{,y}+v_{,x})\}dx\,dy
\end{aligned} \tag{1b}$$

$$P_4{}^0[u] = \frac{3}{2(q_0)^2}\int_0^{\pi\theta}\int_0^{q_0L/R}(w_{,x}{}^2 + w_{,y}{}^2)^2\,dx\,dy \tag{1b (Cont.)}$$

In the foregoing, c_1 and c_2 are found to be [22],

$$\begin{aligned}
c_1 &= [-t/(Rc)]\,[\sigma(1+\alpha_s) + p\nu]/(1+E_n\alpha_s) \\
c_2 &= [t/(Rc)]\,[\sigma\,\nu\,(1+\alpha_s) \\
&\quad + (p)\,(1 + E_n\alpha_s\,(1-\nu^2))]/(1+E_n\alpha_s)
\end{aligned} \tag{2a}$$

and in the important special case of zero lateral pressure and equal Young's moduli,

$$c_1 + \nu c_2 = [t/(Rc)]\,(-\sigma)\,(1-\nu^2) \quad \text{and} \quad c_2 + \nu c_1 = 0 \tag{2b}$$

Furthermore, σ is the axial load (positive for compression) and p is the lateral pressure (positive for internal pressure). Note that unlike the conventional Koiter notation, the applied load is incorporated into the $P_2{}'[u]$ expression. Moreover, integration is only carried out over half of the cylindrical panel between adjacent stringers, where $y = 0$ corresponds to the stringer edge. In addition to the aforementioned, the stringer contribution to the total potential energy must also be included. Thus the potential energy of any one stringer on passing from the prebuckling state to a neighboring state can also be arranged in the form of equation (1) where

$$\begin{aligned}
P_2{}^0[u] &= [1/(2q_0)]\int_0^{q_0L/R}\{(E_n\alpha_s d_s/R)\,[u_{,x} - 2c(e_s/t)w_{,xx}]^2 \\
&+ \beta_s(d_s/R)\,(w_{,xx})^2 + (\gamma/q_0)\,(w_{,xy})^2\}dx \\
P_2{}'[u] &= (1/q_0)E_n\alpha_s(d_s/R)\,(c_1R/t) \\
&\times \int_0^{q_0L/R}\{cw_{,x}{}^2 + 2c^2(e_s/t)^2\,(t/R)\,(w_{,xy})^2\}dx \\
P_3{}^0[u] &= (1/q_0)E_n\alpha_s(d_s/R)\int_0^{q_0L/R}\{[u_{,x} - 2c(e_s/t)w_{,xx}]. \\
&\quad [c(w_{,x})^2 + 2c^2(e_s/t)^2(t/R)\,(w_{,xy})^2]\}dx
\end{aligned} \tag{3}$$

Nomenclature

Dimensional Quantities

A_s = cross-sectional area of a stringer
e_s = stringer eccentricity
$d_s = (2\pi R)/M_s$; stringer spacing
$D = Et^3/(4c^2)$
E, E_s = Young's modulus of the skin and stringer, respectively
F = Airy stress function
G, G_s = shear modulus of the skin and stringer, respectively
I_s, I_t = out-of-plane and tangential moment of inertia of a stringer
J_s = torsional constant of a stringer
L = length of the shell
M_x, M_y, M_{xy} = bending stress resultants
N_x, N_y, N_{xy} = skin stress resultants
$\bar{p}$ = applied lateral pressure, positive for internal pressure
$Q_s = 2\,e_s - t$; height of the stringer defined only for rectangular stringer
R = radius of the shell
t = skin thickness
t_t = tangential thickness of the stringer, defined only for rectangular stringer
U, V = axial and circumferential in-plane displacements, respectively
W = out-of-plane displacement
X, Y = axial and circumferential coordinates, respectively
$\bar{\sigma}$ = applied axial stress

Nondimensional Quantities

a, b = related to the cubic and quartic terms of the potential energy of a single mode system
b_1, b_2 = quartic term of the overall mode and the local panel mode
$e = \sqrt{3(1-\nu^2)}$
$\bar{e}$ = related to the eigenvector of the overall mode
$E_n = E_s/E$
$f = 2cF/(Et^3)$
$h(x, y)$ = amplitude of the local mode which is a slowly varying function of the coordinates
m, n = number of axial half waves and number of circumferential full waves, respectively
M_s = total number of equally spaced stringers
$p = \bar{p}R^2c/(Et^2)$, applied lateral pressure (positive for internal pressure)
$q_0 = \sqrt{(2cR/t)}$
$r = \sigma_c{}^{(1)}/\sigma_c{}^{(2)}$, ratio of the overall to the local buckling load
S_x, S_y = as defined in text
$u, v = (q_0/t)(U, V)$
$w = W/t$
$x, y = (q_0/R)(X, Y)$
$z = [\lambda/(2\theta)]^2$
$Z = [L^2/(Rt)]\sqrt{(1-\nu^2)}$, Batdorf parameter
$\alpha_s = A_s/(d_st)$, area ratio
β_0 = angle in radians between adjacent stringers
$\beta_s = (E_sI_s)/(Dd_s)$, out-of-plane bending stiffness ratio
$\beta_t = E_sI_t/(Dd_s)$, tangential bending stiffness ratio
$\gamma_s = (q_0G_sJ_s)/(DR)$, torsional rigidity ratio
δ/t = amplitude of the local panel mode over the skin thickness
$\bar{\delta}$ = amplitude of local imperfection
$\theta = q_0/M_s$, flatness parameter
λ = ratio of the wavelength of the local mode in the circumferential direction to that in the axial direction
λ^* = as defined in the text
ν = Poisson's ratio
ξ_1 = amplitude of the overall mode with respect to the skin thickness
$\sigma = \bar{\sigma}(Rc/Et)$, nondimensional axial stress relative to the lighter weight unstiffened cylindrical shell
τ = torsional factor which varies from 0.14 for square-shaped stringer to 0.333 for thin-walled rectangular stringer

Note: A comma preceding subscripts indicates differentiation with respect to the variables shown.

$$P_4{}^0[u] = [1/(2q_0)]E_n \alpha_s (d_s/R) \int_0^{q_0 L/R} \{c^2(w_{,x})^4 + [2c^2(e_s/t)^2(t/R)]^2(w_{,xy})^4 + 4c^2(e_s/t)^2(t/R)c(w_{,x})^2(w_{,xy})^2\}dx \quad (3)$$

(*Cont.*)

It should be noted that not all the aforementioned stringer energy contribution is considered by both Koiter and Stephens and it will be shown that, in general, it is highly significant in the computation of the quartic term of the potential energy of the local mode.

Consistent with Koiter's general theory of elastic stability, the total displacements can be expressed as the sum of the prebuckling deformation, the buckling mode, and the second-order field in the following form:

$$\{u_t, v_t, w_t\} = \{u_p, 0, w_p\} + (\delta/t)\{u_c, v_c, w_c\} + (\delta/t)^2\{u_2, v_2, w_2\} \quad (4)$$

where δ/t is the amplitude of the buckling mode normalized with respect to the skin thickness. In the postbuckling regime, it was shown [1, 2] that the equilibrium states are specified by

$$\left(1 - \frac{\sigma}{\sigma_c}\right)(\delta/t) + a(\delta/t)^2 + b(\delta/t)^3 = (\sigma/\sigma_c)(\bar{\delta}/t) \quad (5)$$

where the a and b coefficients are related to the cubic and quartic terms of the potential energy, respectively, $\bar{\delta}$ is the amplitude of the local imperfection which is taken to be of the same form as the buckling mode. In this expression, the cubic term a of the local mode vanishes, provided that the number of axial half waves is even, the number of stringers is even or the shell is infinitely long. Since local buckling of a stiffened shell of sufficiently high Z (>200) generally results in many local buckles, and due to symmetry considerations, the cubic term is assumed to be negligible. Thus a negative value of b indicates that the structure is imperfection-sensitive while a positive value indicates a lack of imperfection-sensitivity. In order to compute the b coefficient, it is necessary to determine the second-order field in addition to the buckling mode. Furthermore, the second-order field (u_2, v_2, and w_2) is obtained by imposing the appropriate orthogonal conditions and by minimizing the quadratic and cubic terms of the potential energy. That is, the minimization

$$\delta P_2{}^\lambda[u_2] + \delta P_3{}^0[u_c] = 0$$

yields the following three-differential equations [1, 19],

$$u_{2,xx} + \tfrac{1}{2}(1-\nu)u_{2,yy} + \tfrac{1}{2}(1+\nu)v_{2,xy} + \nu w_{2,x} = -c\{2w_{c,x}w_{c,xx} + (1+\nu)w_{c,xy}w_{c,y} + (1-\nu)w_{c,x}w_{c,yy}\}$$

$$\tfrac{1}{2}(1+\nu)u_{2,xy} + v_{2,yy} + \tfrac{1}{2}(1-\nu)v_{2,xx} + w_{2,y} = -c\{2w_{c,y}w_{c,yy} + (1+\nu)w_{c,x}w_{c,xy} + (1-\nu)w_{c,xx}w_{c,y}\}$$

$$(u_{2,x})\left(\frac{\nu}{1-\nu^2}\right) + (v_{2,y})\left(\frac{1}{1-\nu^2}\right) + \nabla^4 w_2 + 2S\,w_{2,xx} - 2p\,w_{2,yy} + (w_2)\left(\frac{1}{1-\nu^2}\right) = 2c\left\{f_{c,yy}w_{c,xx} + f_{c,xx}w_{c,yy} + (-2)f_{c,xy}w_{c,xy} - \left(\frac{\nu w_{c,x}{}^2 + w_{c,y}{}^2}{2(1-\nu^2)}\right)\right\} \quad (6)$$

where S is defined to be

$$S = (\sigma)[(1+\alpha_s)/(1+E_n\alpha_s)] - (p\nu)[(E_n\alpha_s)/(1+E_n\alpha_s)] \quad (7)$$

and it should be noted that $S = \sigma$ in the important special case ($p = 0, E_n = 1, \alpha_s \neq 0$) or in the case $\alpha_s = 0$. Furthermore, the Airy stress function f_c is related to u_c, v_c, and w_c by

$$(1-\nu^2)(f_{c,xx}) = \nu u_{c,x} + (v_{c,y} + w_c)$$

$$(1-\nu^2)(f_{c,yy}) = u_{c,x} + \nu(v_{c,y} + w_c)$$

$$-2(1+\nu)(f_{c,xy}) = u_{c,y} + v_{c,x} \quad (8)$$

In addition to the continuity requirements of the displacements at the stringer edge, i.e., $u^+ = u^-$, $v^+ = v^-$, $w^+ = w^-$, and $w^+_{,y} = w^-_{,y}$, the four jump conditions at the stringer edge (presented in dimensional form for clarity purposes) are given by

$$U = 0 \quad \text{or} \quad N^+_{xy} - N^-_{xy} + (E_sA_s)(\epsilon^+_x - e_s W^+_{,XX} + \tfrac{1}{2}e_s{}^2 W^+_{,XY}{}^2)_{,X} = 0$$

$$V = 0 \quad \text{or} \quad N^+_y - N^-_y - E_sI_t V^+_{,XXXX} = 0$$

$$W = 0 \quad \text{or} \quad (-D)(W^+_{,YYY} - W^-_{,YYY}) + e_sN^+_{s,XX} - E_sI_sW^+_{,XXXX} + N^+_s W^+_{,XX} = 0$$

$$W_{,Y} = 0 \quad \text{or} \quad D(W^+_{,YY} - W^-_{,YY}) + G_sJ_sW^+_{,XXY} + e_s{}^2 N^+_s W^+_{,XXY} + e_s{}^2 N^+_{s,X} W^+_{,XY} = 0 \quad (9a)$$

where subscripts "+" and "−" denote the positive and negative sides of a stringer as measured in the circumferential direction. Moreover, N_s is the axial force applied at the centroid of the stringer and it is given by

$$N_s = (E_sA_s)(\epsilon_x - e_sW_{,XX} + \tfrac{1}{2}e_s{}^2W_{,XY}{}^2) \quad (9b)$$

The desired jump conditions for the buckling state and for the second-order field computations are obtained by substituting $u_t = u_p + (\delta/t)\,u_c(x,y) + (\delta/t)^2\,u_2(x,y)$ into the foregoing conditions and collecting terms involving (δ/t) and $(\delta/t)^2$, respectively. They will be presented subsequently.

Classical Local Buckling Load

As the applied load is increased from zero, the classical buckling load is defined to be the smallest eigenvalue which causes the quadratic terms of the potential energy to vanish. Due to symmetry considerations, the local buckling mode $u_c(x,y)$, $v_c(x,y)$, $w_c(x,y)$ must be composed of either even or odd functions with respect to the stringer edge. It is found in the example problem that the smallest eigenvalue corresponds to the condition that $u_c(x,y)$ and $w_c(x,y)$ are odd functions while $v_c(x,y)$ is an even function with respect to the stringer edge. This implies that $u_c(x, y=0) = w_c(x, y=0) = 0$ and thus the corresponding jump conditions can be neglected. If the shell is assured to be sufficiently long such that there are many local waves in the axial direction, then the buckling mode is expressible in the form

$$u_c(x,y) = u_c(y)\cos[\lambda x/(2\theta)]$$

$$v_c(x,y) = v_c(y)\sin[\lambda x/(2\theta)]$$

$$w_c(x,y) = w_c(y)\sin[\lambda x/(2\theta)] \quad (10)$$

The boundary conditions at the stringer edge are

$$u_c(y=0) = 0$$

$$w_c(y=0) = 0$$

$$v_c(y=0)_{,y} - \pi(1-\nu^2)(\beta_t)(\theta/q_0{}^2)(\lambda/2\theta)^4 v_c(y=0) = 0$$

$$[\lambda^2/(8\theta^3)](\gamma_s)w_c(y=0)_{,y} - w_c(y=0)_{,yy} = 0 \quad (11)$$

where β_t is the nondimensional stringer tangential bending stiffness ratio and γ_s is related to the torsional rigidity of the stringer. Since the tangential bending energy of the stringer is usually small compared with the out-of-plane bending energy and the purpose of the present paper is to study the effect of stringer eccentricity and cross-sectional area on the imperfection-sensitivity of the cylindrical panel, the quantity β_t is set to zero for comparison purposes with Stephens' paper [18]. Making such a simplification, the present eigenvalue problem is identical to that examined by Stephens. In the computation of the buckling load, the mixed formulation is used instead of the UVW formulation due to its simplicity.

The ordinary differential equations are

$$w_c(y)_{,yyyy} - 2(z+p)w_c(y)_{,yy} + (z^2 - 2zS)w_c(y) - zf_c(y) = 0$$

$$f_c(y)_{,yyyy} - 2zf_c(y)_{,yy} + z^2f_c(y) + zw_c(y) = 0 \quad (12)$$

where $z = [\lambda/(2\theta)]^2$ and S is defined in equation (7). The four boundary conditions at the stringer edge are

$$w_c(y=0) = 0, \quad (z/2)\gamma_s w_c(y=0)_{,y} - w_c(y=0)_{,yy} = 0,$$

and

$$f_c(y=0) = 0, \quad f_c(y=0)_{,yy} = 0 \tag{13}$$

while the boundary conditions at the midpanel are

$$w_c(y=\pi\theta)_{,y} = 0, \quad w_c(y=\pi\theta)_{,yyy} = 0,$$

and

$$f_c(y=\pi\theta)_{,y} = 0, \quad f_c(y=\pi\theta)_{,yyy} = 0 \tag{14}$$

The foregoing ordinary differential equations with boundary conditions can be solved using a central finite-difference scheme in conjunction with the computation of the determinants as was done by Stephens and others using a modification of Potter's method [21]. However, since only the smallest eigenvalue is of interest, it is felt that the previous method can be replaced in the following way by a more efficient technique. First, the resulting system of linear finite-difference equations were rearranged into the standard form $Ax = \sigma Bx$ where A and B are both banded matrices and B is singular (the matrix B should not be confused with the interaction parameter B employed in [14]). The shifted inverse power method was then used to compute the eigenvector and eigenvalue, details of which can be found in [22, 23]. This obviates the need to compute determinants and drastically reduces the computation time. Second, a more efficient linear equation solver [20] was used instead of Potter's method. The classical buckling load was found by computing the smallest eigenvalue for all possible wavelengths λ (usually the optimum λ ranges between one and two). It was observed that the quartic term measured by the b coefficient, was considerably more sensitive to the wavelength parameter than was the buckling load. Consequently, 61 finite-difference grid points were employed instead of the 30 stations used by Stephens [18]. Nevertheless, the convergence is so rapid that it usually requires very few iterations.

Second-Order Fields of the Local Mode

Upon substituting $w_c(x, y), f_c(x, y)$ into equation (6), the solutions for u_2, v_2, and w_2 can be expressed in the form

$$\begin{aligned} u_2(x,y) &= (x)(u_0) + u_\beta(y)\sin(\lambda x/\theta), \\ v_2(x,y) &= v_\alpha(y) + v_\beta(y)\cos(\lambda x/\theta) \\ w_2(x,y) &= w_0 + w_\alpha(y) + w_\beta(y)\cos(\lambda x/\theta) \end{aligned} \tag{15}$$

Consequently, the partial differential equations can be decoupled into two sets of ordinary differential equations in terms of $v_\alpha(y)$, $w_\alpha(y)$ and $u_\beta(y)$, $v_\beta(y)$, $w_\beta(y)$, respectively. For brevity, they are called the alpha and beta problems. The differential equations for the alpha problem are then given by

$$v_\alpha(y)_{,yy} + w_\alpha(y)_{,y} = (-c)[w_c(y)_{,y}w_c(y)_{,yy} + \nu z w_c(y)w_c(y)_{,y}]$$

and

$$\begin{aligned} w_\alpha(y)_{,yyyy} - 2p\, w_\alpha(y)_{,yy} + \left(\frac{1}{1-\nu^2}\right)[w(y)] + \left(\frac{1}{1-\nu^2}\right)[v_\alpha(y)] \\ = c[-zw_c(y)f_c(y)_{,yy} - zw_c(y)_{,yy}f_c(y) - 2zw_c(y)_{,y}f_c(y)_{,y}] \\ - \left(\frac{c}{2(1-\nu^2)}\right)[\nu z w_c(y)^2 + w_c(y)_{,y}{}^2] - \left(\frac{1}{1-\nu^2}\right)(\nu u_0 + w_0) \end{aligned} \tag{16}$$

From symmetry considerations, $v_2(x, y = 0) = 0$ and $w_2(x, y = 0)_{,y} = 0$ and hence the corresponding jump conditions can be neglected. Thus, for the alpha problem, the jump condition with respect to w_2 implies that $w_\alpha(y = 0)_{,yyy} = 0$. The boundary conditions at the midpanel due to symmetry are: $v_\alpha(y = \pi\theta) = 0$, $w_\alpha(y = \pi\theta)_{,y} = 0$, $w_\alpha(y = \pi\theta)_{,yyy} = 0$. In a similar manner, the ordinary differential equations for the beta problem are

$$\begin{aligned} [-(\lambda/\theta)^2 u_\beta(y) + \tfrac{1}{2}(1-\nu)u_\beta(y)_{,yy}] - \tfrac{1}{2}(1+\nu)(\lambda/\theta)v_\beta(y)_{,y} \\ - \nu(\lambda/\theta)w_\beta(y) = \left(\frac{c}{2}\right)\left(\frac{\lambda}{2\theta}\right)\{(2z)[w_c(y)]^2 - (1+\nu)[w_c(y)_{,y}]^2 \\ - (1-\nu)w_c(y)w_c(y)_{,yy}\} \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}(1+\nu)\left(\frac{\lambda}{\theta}\right)u_\beta(y)_{,y} + [v_\beta(y)_{,yy} \\ - \tfrac{1}{2}(1-\nu)(\lambda/\theta)^2 v_\beta(y)] + w_\beta(y)_{,y} = (-c)[-w_c(y)_{,y}w_c(y)_{,yy} \\ + (z)w_c(y)w_c(y)_{,y}] \end{aligned}$$

$$\begin{aligned} \left[\left(\frac{\nu}{1-\nu^2}\right)(\lambda/\theta)u_\beta(y)\right] + \left(\frac{1}{1-\nu^2}\right)[v_\beta(y)_{,y}] \\ + \Bigg\{w_\beta(y)_{,yyyy} + [-2(\lambda/\theta)^2 - 2p][w_\beta(y)_{,yy}] + \Bigg[(\lambda/\theta)^4 - (2S)(\lambda/\theta)^2 \\ + \left(\frac{1}{1-\nu^2}\right)\Bigg] w_\beta(y)\Bigg\} = (c)\,[zw_c(y)f_c(y)_{,yy} + zw_c(y)_{,yy}f_c(y) \\ -2\,zw_c(y)_{,y}f_c(y)_{,y}] + [c/(2(1-\nu^2))]\,[-\nu\, zw_c(y)^2 + w_c(y)_{,y}{}^2] \end{aligned} \tag{17}$$

Again, using symmetry at the stringer edge, $v_\beta(y = 0) = 0$ and $w_\beta(y = 0)_{,y} = 0$ and the corresponding jump conditions with respect to v_2 and $w_{2,y}$ may be neglected. The jump conditions with respect to u_2 and w_2 are

$$\begin{aligned} u_\beta(y=0)_{,y} - \{[(E_n\alpha_s)\,(2\pi\theta)\,(1+\nu)\,(\lambda/\theta)^2] \\ \times [u_\beta(y=0) + (2c)\,(e_s/t)\,(\lambda/\theta)\,w_\beta(y=0) \\ + (c^2/4)\,(t/R)\,(e_s/t)^2\,(\lambda/\theta)\,w_c(y=0)_{,y}{}^2]\} = 0 \end{aligned} \tag{18}$$

$$\begin{aligned} w_\beta(y=0)_{,yyy} + [c/(1+\nu)]\,(e_s/t)\,(\lambda/\theta)\,u_\beta(y=0)_{,y} \\ + [\beta_s\pi\,\theta(\lambda/\theta)^4 + (E_n\alpha_s)(c)\,(c_1)\,(2\pi R/t) \\ \times \theta(\lambda/\theta)^2]\,w_\beta(y=0) = 0 \end{aligned} \tag{19}$$

Again, from symmetry conditions, the four boundary conditions at the midpanel are $u_\beta(y = 0)_{,y} = 0$, $v_\beta(y = 0) = 0$, $w_\beta(y = 0)_{,y} = 0$ and $w_\beta(y = 0)_{,yyy} = 0$. The uncoupled alpha and beta problems are then solved using a central finite-difference scheme.

Note that values for the constants u_0 and w_0 [equation (15)] cannot be obtained from the jump conditions but are obtained by minimizing

$$(P_2[u_2] + P_{21}[u_c, u_2])_{\text{skin}} + (P_2[u_2] + P_{21}[u_c, u_2])_{\text{stringer}} \tag{20}$$

with respect to u_0 and w_0. After some algebra, they are found as [22],

$$\begin{aligned} u_0 &= (R_1 - R_2)/[(1-\nu^2)\,(1 + E_n\alpha_s)] \\ w_0 &= \{[1 + (1-\nu^2)(E_n\alpha_s)](R_2) - \nu R_1\}/[(1-\nu^2)\,(1 + E_n\alpha_s)] \end{aligned} \tag{21a}$$

where R_1 and R_2 are defined by

$$\begin{aligned} R_1 &= (-c/2)\,[z\,H_1 + \nu H_2] - [(1-\nu^2)\,(E_n\alpha_s)] \\ &\quad \times [(c^2)\,(e_s/t)^2(t/R)(z)w_c(y=0)_{,y}{}^2] \\ R_2 &= (-c/2)\,[z\nu H_1 + H_2] \\ H_1 &= [1/(\pi\theta)]\int_0^{\pi\theta}\{w_c(y)^2\}dy, \\ H_2 &= [1/(\pi\theta)]\int_0^{\pi\theta}\{w_c(y)_{,y}{}^2\}dy \end{aligned} \tag{21b}$$

In the special case of zero torsional rigidity, one obtains,

$$H_1 = 1/2, \quad H_2 = 1/(8\theta^2), \quad w_c(y=0)_{,y} = 1/(2\theta), z = [\lambda/(2\theta)]^2 \tag{21c}$$

so that

$$u_0 = [1/(1 + E_n \alpha_s)] [-c/(16\,\theta^2)][1 + (E_n \alpha_s)(e_s/t)^2(t/R)(c/\theta^2)]$$

$$w_0 = [1/(1 + E_n \alpha_s)][-c/(16\theta^2)]\{1 + (E_n \alpha_s) - (c/\theta^2)(e_s/t)^2(t/R)\}\} \quad (22)$$

It can be seen that the constants u_0 and w_0 converge to Koiter's result [17] for no applied pressure, $\lambda = 1$, $e_s/t = 0$, and $\alpha_s = 0$.

Initial Postbuckling Behavior

The potential energy expression of a single mode symmetric system, evaluated at the classical buckling load is (note that the geometric initial imperfection is taken to be of the same shape as the buckling mode)

$$P.E./(Et^3) = (A_4)(\delta/t)^4 + (\sigma - \sigma_c)(d)(\delta/t)^2 + (2d\sigma)(\delta/t)(\bar{\delta}/t) \quad (23a)$$

where $(\bar{\delta}/t)$ is the amplitude of the imperfection. Moreover, in the present stringer reinforced cylindrical panel problem, A_4 and d are computed from

$$A_4 = (P_4{}^0[u_c] - (P_2{}^0[u_2] + P_2{}'[u_2]))_{\text{skin}} + (P_4{}^0[u_c] - (P_2{}^0[u_2] + P_2{}'[u_2]))_{\text{stringer}}$$

$$d = \left(\frac{\partial}{\partial\sigma} P_2{}'\,[u_c]\right)_{\text{skin}} + \left(\frac{\partial}{\partial\sigma} P_2{}'[u_c]\right)_{\text{stringer}} \quad (23b)$$

Thus the equilibrium equation can be written as,

$$b(\delta/t)^3 + [1 - (\sigma/\sigma_c)](\delta/t) = (\sigma/\sigma_c)(\bar{\delta}/t) \quad (24a)$$

where the b coefficient of the present local problem is

$$b = (4A_4)/[(-2d)(\sigma_c)] \quad (24b)$$

The structure is imperfection-sensitive if the b coefficient is negative. Since the buckling mode u_c and the second-order field u_2 are now known quantities, it is straightforward to compute the individual terms of the b coefficient. The quartic term of the potential energy for the skin (from stringer edge to the midpanel) is

$$\{P_4{}^0[u_c] - (P_2{}^0[u_2] + P_2{}'[u_2])\}_{\text{skin}} = Q_1 - (Q_2 + Q_3 + Q_4) \quad (25a)$$

where

$$Q_1 = P_4{}^0[u_c]_{\text{skin}} = [1/(1 - \nu^2)](c/4)(t/R)[q_0 L/(8R)] \times \int_0^{\pi\theta} \{[(3\lambda^4)/(16\theta^4)]w_c(y)^4 + 3w_c(y)_{,y}{}^4 + [\lambda^2/(2\theta^2)][w_c(y)^2 w_c(y)_{,y}{}^2]\}dy$$

$$Q_2 = \left(\frac{1}{2(1-\nu^2)}\frac{L}{q_0 R}\right)\int_0^{\pi\theta}\{[w_\alpha(y) + v_\alpha(y)_{,y}]^2 + [w_\alpha(y) + v_\alpha(y)_{,y}](2w_0 + 2\nu u_0) + (c^2/3)[w_\alpha(y)_{,yy}{}^2] + q_0{}^2(c_2 + \nu c_1)[w_\alpha(y)_{,y}]^2\}\,dy$$

$$Q_3 = \left(\frac{1}{4(1-\nu^2)q_0{}^2}\right)\left(\frac{q_0 L}{R}\right)\int_0^{\pi\theta}\{(\lambda/\theta)^2[u_\beta(y)]^2 + [w_\beta(y) + v_\beta(y)_{,y}]^2 + 2\nu(\lambda/\theta)u_\beta(y)[w_\beta(y) + v_\beta(y)_{,y}] + \tfrac{1}{2}(1-\nu)[u_\beta(y)_{,y} - (\lambda/\theta)v_\beta(y)]^2 + (c^2/3)[(\lambda/\theta)^4 w_\beta(y)^2 + w_\beta(y)_{,yy}{}^2 - 2\nu(\lambda/\theta)^2\, w_\beta(y)w_\beta(y)_{,yy} + 2(1-\nu)(\lambda/\theta)^2[w_\beta(y)_{,y}]^2] + q_0{}^2[(c_1 + \nu c_2)(\lambda/\theta)^2 w_\beta(y)^2 + (c_2 + \nu c_1)[w_\beta(y)_{,y}{}^2]]\}\,dy$$

$$Q_4 = \left[\frac{1}{4(1-\nu^2)}\frac{\beta_0 L}{R}\right](u_0{}^2 + w_0{}^2 + 2\nu u_0 w_0) \quad (25b)$$

In addition, the stringer contribution to the quartic term is (upon dividing the result by two since only half a cylindrical panel between adjacent stringers has been considered)

$$\{P_4{}^0[u_0] - (P_2{}^0[u_2] + P_2{}'[u_2])\}_{\text{stringer}} = S_1 - S_2$$

where

$$S_1 = P_4{}^0[u_c]_{\text{stringer}} = \tfrac{1}{4}E_n\alpha_s(d_s/R)(L/R) \times [3/(32\theta^4)]c^4(e_s/t)^4(t/R)^2\lambda^4[w_c(y=0)_{,y}]^4$$

$$S_2 = P_2{}^0[u_2]_{\text{stringer}} = \tfrac{1}{4}E_n\alpha_s(d_s/R)(L/R) \times [u_0{}^2 + \tfrac{1}{2}(\lambda/\theta)^2\, u_\beta(y=0)^2] \quad (26)$$

Also for the skin,

$$P_2{}'[u_c]_{\text{skin}} = (\sigma_c)\left[(-1/16)(q_0 L/R) \times (t/Rc)(\lambda/\theta)^2\int_0^{\pi\theta} w_c(y)^2\,dy\right] \quad (27a)$$

and for the stringer (again, upon dividing by two),

$$P_2{}'[u_c]_{\text{stringer}} = (\sigma_c)\,[(-t/Rc)E_n\alpha_s(e_s/t)^2c^2(d_s/R)z \times [w_c(y=0)_{,y}]^2\cdot\tfrac{1}{2}\cdot(L/R)] \quad (27b)$$

Mode Interaction by Amplitude Modulation of the Local Mode

In order to investigate imperfection-sensitivity due to local and overall mode interactions, Koiter's theory of amplitude modulation of the local mode [14] was used to compute the two-mode potential energy expression. This necessitates incorporating the appropriate terms to account for axial stiffness, torsional rigidity, and eccentricity of the stringers. For brevity, only the final form of the potential energy expression is given (for details, refer to [22]),

$$\frac{P.E.}{(Et)(2\pi RL)} = [1 - (\lambda^*/r)](a_1)^2 + (1-\lambda^*)\iint h(x,y)^2\,ds + (C_1/2)(a_1)^4 + (b_2/2)\iint h(x,y)^4\,dS + (2B)(a_1)\iint\{(\sqrt{2})\,[\sin(m\pi X/L)\sin(nY/R)]h(x,y)^2\}\,dS - (2)\,(\lambda^*/r)(\bar{a}_1 a_1) - (2\lambda^*)\iint \bar{h}(x,y)\,h(x,y)\}\,dS \quad (28)$$

where $dS = (dX\,dY)/(2\pi RL)$, $h(x, y)$ is the modulated amplitude of the local mode normalized with respect to the skin thickness, barred quantities refer to imperfections, b_2 is the b coefficient of the local mode. The overall mode [15] and the modulated local mode are

$$\{W_c{}^{(1)}, F_c{}^{(1)}\} = \xi_1\{t, Et^3\bar{e}\}\sin(m\pi X/L)\sin(nY/R) \quad (29a)$$

$$W_c{}^{(2)} = (t)\,h(x,y)\,[\sin(\lambda\pi X/d_s)\,w_c(y)] \quad (29b)$$

Furthermore, a_1, $\bar{a}_1$, B, C_1, λ^*, and r are defined to be

$$a_1 = (\xi_1)\sqrt{(Z_1/Z_2)},\quad \bar{a}_1 = (\bar{\xi}_1)\sqrt{(Z_1/Z_2)}$$

$$B = Z_3/[(2)\,(\sqrt{Z_1Z_2})],\quad C_1 = (Z_2/Z_1)(b_1)$$

$$\lambda^* = \sigma/\sigma_c{}^{(2)},\quad r = \sigma_c{}^{(1)}/\sigma_c{}^{(2)},\quad \sigma/\sigma_c{}^{(1)} = \lambda^*/r \quad (30)$$

where $\sigma_c{}^{(1)}$ and $\sigma_c{}^{(2)}$ are the overall and local buckling loads, respectively, b_1 is the b coefficient of the overall mode. In the foregoing expressions, Z_1, Z_2, and Z_3 are defined as

$$Z_1 = [\sigma_c{}^{(1)}/(8c)][(1+\alpha_s)(m\pi)^2(t/R)^3\,(R/L)^2]$$

$$Z_2 = \sigma_c{}^{(2)}\,[H_1/(16c)](t/R)^3(\lambda M_s)^2$$

$$Z_3 = [1/(4\sqrt{2})]\,(t/R)^2(M_s)^2[(S_x\lambda^2)(H_1/4) + S_y\theta^2 H_2] \quad (31)$$

where H_1 and H_2 are defined in equations (21b) and (21c) and S_x, S_y are (using classical simple support conditions at $X = 0, L$),

$$S_x = [-A_{xx}(m\pi)^2(t/L)^2 - A_{xy}n^2(t/R)^2](\bar{e}) + [-(\overline{B}_{xx})(m\pi)^2(t/L)^2 - \overline{B}_{xy}n^2(t/R)^2]$$

$$S_y = [-A_{yy}n^2(t/R)^2 - A_{yx}(m\pi)^2(t/L)^2](\bar{e}) + [(-\overline{B}_{yy})n^2(t/R)^2 - \overline{B}_{yx}(m\pi)^2(t/L)^2] \quad (32)$$

and A_{xx}, A_{xy}, A_{yx}, A_{yy}, $\overline{B}_{xx}$, $\overline{B}_{xy}$, $\overline{B}_{yx}$, and $\overline{B}_{yy}$ can be obtained from Hutchinson and Amazigo's paper [15].

Finally, in the presence of only a local imperfection, the bifurcation load (due to the influence of the local imperfection triggering bifur-

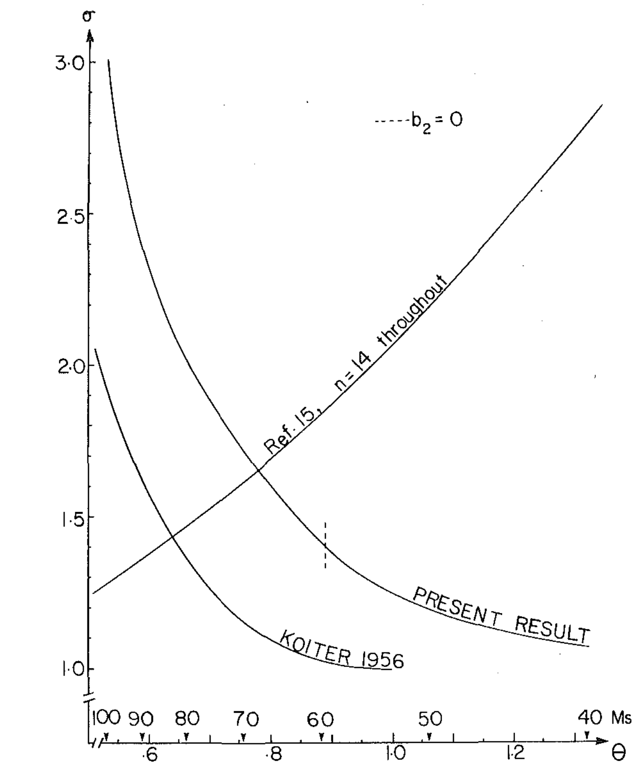

Fig. 1 Classical buckling load versus flatness parameter

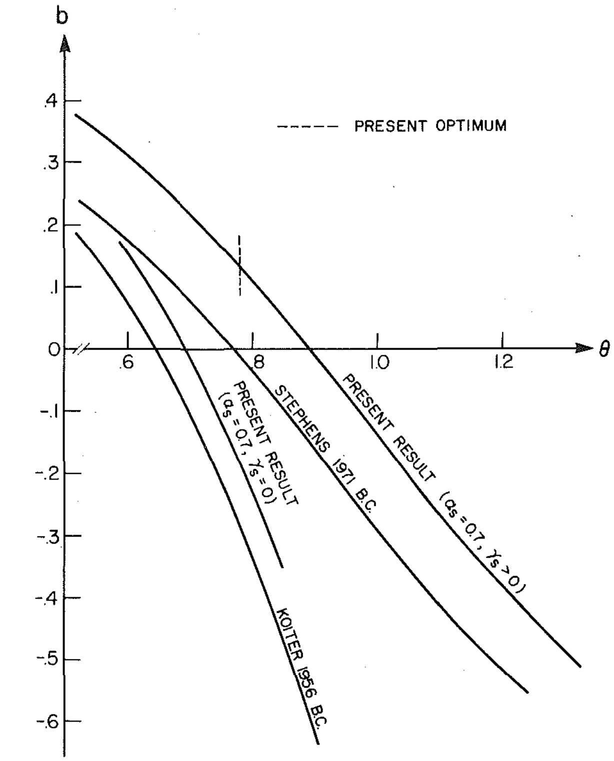

Fig. 2 *b* coefficient of the local mode versus flatness parameter

cation into the overall mode) of the imperfect system $\lambda_c{}^*$ is related to the amplitude of the local imperfection h_0 by [22]

$$(1-\lambda_c{}^*)\left(\frac{2B^2-2b_2(1-\lambda_c{}^*/r)}{2B^2-3b_2(1-\lambda_c{}^*/r)}\right)\left(\frac{(1-\lambda_c{}^*)(1-\lambda_c{}^*/r)}{2B^2-3b_2\,(1-\lambda_c{}^*/r)}\right)^{1/2} = (\lambda_c{}^*)\,(\pm h_0) \quad (33)$$

In the event that the b coefficient of the local panel mode turns out to be positive, the previous relation yields a lower bound given by [14]

$$L.B. = (r)\{1-[(2B^2)/(3b_2)]\} \quad (34)$$

Example Problem

To illustrate the effects of stringer eccentricity, axial stiffness, and torsional rigidity on buckling, imperfection-sensitivity, and mode interaction, example problems have been chosen from the relevant published literature [6, 9, 14, 24, 25] details of which can be found in [22]. However, for brevity, only one example will be presented here based on the following shell parameters taken from Byskov and Hutchinson [6]. The fixed parameters are

$$\alpha_s = 0.7, \quad Z = 810.848, \quad R/t = 850, \quad \nu = 0.3, \quad q_0 = 52.998$$
$$R/L = 1.0, \quad E_n = 1.0, \quad t_t/t = 4.09, \quad \beta_t = 10.6558. \quad (35)$$

Note that since the stringers are constrained to be of rectangular shape and α_s, t_t/t are held constant, then changing the number of stringers requires changes in the stringer height Q_s as well as

$$\theta = q_0/M_s, \quad Q_s/t = [\alpha_s/(t_t/t)]\,(2\pi/M_s)(R/t), \quad e_s/t = (1/2)\,[1+(Q_s/t)]$$
$$\beta_s = (E_n\alpha_s)\,(c^2/3)\,(Q_s/t), \quad d_s/t = (t_t/t)\,(Q_s/t)\,(1/\alpha_s) \quad (36)$$

The torsional rigidity ratio can be obtained from

$$\gamma_s = [(12\pi E_n\theta)(\tau)\text{ minimum }(\beta_s,\beta_t)/(1+\nu)] - (E_n\alpha_s)(e_s/t)^2(8\pi c_1)(c^2\theta) \quad (37)$$

where τ is the torsional constant which ranges from 0.14 for square shape stringers to 0.333 for thin-walled stringers. The second term in equation (37) refers to the axial shortening of the stringer due to twisting and is usually quite small compared with the first term.

Using these values, the classical buckling load normalized with respect to the lighter weight unstiffened cylindrical shell classical buckling load is plotted against the flatness parameter θ and number of stringers M_s in Fig. 1. It should also be pointed out that this analysis differs from that of Stephens since the permissible wave number λ is treated as discrete and axial shortening of the stringers due to twisting is included. However, buckling loads close to Stephens' result were obtained, thus substantiating that significant increases in local panel buckling loads arise primarily from the inclusion of stringer torsional rigidity. Based on the smeared-out overall buckling analysis of [15] (classical simple support, $X = 0, L$), it can be seen in Fig. 1 that the simultaneous occurrence of local and overall buckling occurs at $\theta = 0.78$ ($M_s = 68$) with $\sigma = 1.66$ instead of $\theta = 0.64$ ($M_s = 83$) with $\sigma = 1.43$ found in [6, 24].

When the variation in the local b coefficient with θ (Fig. 2) is considered, the effect of stringer eccentricity and axial stiffness shows essentially a constant shift from Stephens' result [18]. This is due to the fact that the area ratio α_s is kept constant and that the percentage transfer of the axial stress from the skin to the stringer is a function of α_s. By examining the change in the axial strain via the constant u_0 in equation (15), an approximate formula for the aforementioned percentage transfer of the axial stress is [22],

$$\text{percent transfer} = [\alpha_s/(1+\alpha_s)]\,(100\text{ percent}) \quad (38)$$

Moreover, the transition between stable and unstable postbuckling behavior is indicated by the sign change in the b coefficient which occurs at $\theta = 0.89$ (with $M_s = 60$) rather than at $\theta = 0.775$ (with $M_s = 68$) which is predicted by Stephens [18]. From Fig. 1, it may be seen that coincident buckling occurs at $\theta = 0.78$ which corresponds to $b = 0.137$. Thus the local mode alone is postbuckling stable. It is important to distinguish the b coefficients discussed in [17, 18] (which refer to the quartic term of the skin only) and the b coefficient of the present analysis (which refers to the quartic term of the entire skin and stringer combination). For example, in Fig. 2 this difference can be

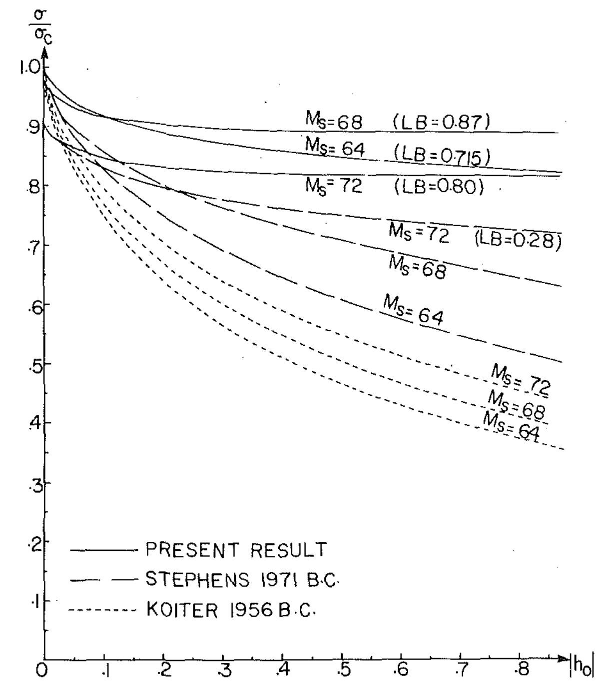

Fig. 3 Bifurcation load normalized with respect to $\sigma_c{}^{(2)}$ versus local imperfection amplitude normalized with respect to the skin thickness

seen by first comparing the present results for $\alpha_s = 0.7, \gamma_s = 0$, with those of Koiter [17]. The shift in curves is due to the inclusion of the axial stiffness parameter α_s (and stringers) in calculating the postbuckling redistribution of the compressive load. As predicted by Koiter, the difference is rather small. However, when the torsional stiffness is also considered ($\gamma_s > 0$), it is evident that very large changes in the b coefficient occur.

At this point, it is of particular interest to study the effect of a nonlinear prebuckling state (due to the presence of a local imperfection alone) on the previous local and overall mode interaction problem. The four parameters required to specify the two-mode potential energy expression are: the transformed b coefficient of the overall mode C_1, the b coefficient of the local mode b_2, the interaction cubic term B, and the ratio of the overall to the local buckling load r. Based on the computation of these parameters for the three designs being considered (M_s = 64, 68, and 72), the bifurcation load normalized with respect to the classical buckling load of the local mode $\sigma/\sigma_c{}^{(2)}$ (it should be cautioned that $\sigma_c{}^{(2)}$ differs between the present and Koiter's 1956 analyses) is plotted against the amplitude of the local imperfection in Fig. 3. Comparing these solutions, one can conclude that significant differences arise due to the inclusion of axial stiffness and torsional rigidity of the stringers. The present curves tend to "level off" quite rapidly for imperfection amplitudes likely to be encountered in practice. Table 1 summarizes the essential results together with the lower bound estimates (from equation (34)) when b_2 is positive. Furthermore, it should be noted that the value of the b coefficient of the overall mode has no effect whatsoever in Fig. 3 (see [14]). It was also found that each circumferential half wave of the overall mode involved about two to three stringers, thus indicating that the smeared-out overall mode analysis is a reasonable approximation for this case.

Finally, it should be emphasized that several other example problems based on different shell configurations have been examined. It was found that all these investigations yield qualitatively similar results. Experimental results confirming the present buckling and initial postbuckling behavior of the local mode can be found in [25].

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Table 1 Summary of mode interaction parameters

M_s	θ	r	B	b_2 (present)	b_2 (Stephens)	Lower bound (present)	Lower bound (Stephens)
64	0.828	1.144	0.158	0.044	−0.0922	0.715	No *L.B.*
68	0.779	1.0045	0.1665	0.137	−0.00457	0.8695	No *L.B.*
72	0.736	0.8842	0.1741	0.211	0.064	0.799	0.2788

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