Effects of Geometric Imperfections on Vibrations of Biaxially Compressed Rectangular Flat Plates

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This paper deals with the effects of geometric imperfections on the vibration frequencies of simply supported flat plates under in-plane uniaxial or biaxial compression. The analysis is based on a solution of the nonlinear von Kármán equations for finite deflections, incorporating the influence of an initial geometric imperfection. It is found that significant increase in the vibration frequencies may occur for imperfection amplitude of the order of a fraction of the plate thickness, even in the absence of in-plane forces.

1 Introduction

Vibrations of thin-walled structures is a topic that has received considerable attention over the past decades and continues to remain of major concern because of its importance as a critical design consideration in many mechanical, aerospace, and hydrospace structural applications. Extensive efforts have been devoted to the investigations of vibrations of flat plates and perfect shells of various types of shapes, loadings, boundary conditions, and initial preload [1-3]. Although it is well established that the main reason for the discrepancy between the theoretical and experimental buckling loads for shells is attributed to the presence of initial geometric imperfections, the effects of imperfections on the natural vibrational frequencies of shell structures has received relatively little attention. In fact, among hundreds of theoretical investigations on the vibrations of shells listed in [2], none of them deals with the influence of initial geometric imperfections. Since then, Rosen and Singer [4, 5], Watawala [6], Singer and Prucz [7], and Hui and Leissa [8] have shown that imperfections of the order of shell thickness may have a pronounced effect on the vibrations of isotropic cylindrical and spherical shells. Effects of geometric imperfections on large amplitude vibrations of circular [9] and rectangular plates [10], cylindrical panels [11], and shallow spherical shells [12] were also examined. Surprisingly, the effect of imperfections on the vibrations of flat plates under initial in-plane stress has not been previously investigated in the literature.

The present study deals with the effects of geometric imperfections on the small amplitute vibrations of simply supported rectangular flat plates with the possibility of in-plane uniaxial or biaxial preload. The imperfections are taken to be a sinusoidal wave in both the in-plane directions. The analysis is based on a solution of the von Kármán equations for finite deflections within the context of Koiter's special theory of elastic stability [13]. The previbration state of static equilibrium is solved by satisfying the nonlinear compatibility equation exactly. The nonlinear equilibrium equation is then satisfied approximately using the Galerkin procedure. The applied load is assumed to be fixed and the appropriate differential equations for the vibration state are obtained by perturbing the previbration state and linearizing the governing differential equations for the vibration state are obtained by perturbing the previbration state and linearizing the governing differential equations with respect to the perturbed quantities. Based on the assumed vibration mode shape, the exact solution for the compatibility equation is found while the Galerkin procedure is used to satisfy the dynamic equilibrium equation approximately. The frequencies are minimized for all possible discrete vibrational wave numbers and geometric imperfections of various wave numbers are considered in the analysis. Since both the previbration and vibration compatibility equations are identically satisfied, the present analysis will yield upper bound frequencies. It is found that the vibration frequencies may be significantly higher than those for perfect plates (the sensitivity to imperfections increase with the amount of preload) for imperfection amplitudes of the order of a fraction of the plate thickness.

2 Governing Differential Equations

The governing nonlinear dynamic analogue of von Kármán equilibrium and compatibility equations for an isotropic, homogeneous, flat plate [14] may be generalized to incorporate the possibility of an initial geometric imperfection W_0 written in terms of the normal displacement W and a stress function F as

Contributed by the Applied Mechanics Division and presented at the Winter Annual Meeting, Boston, Mass., November 13-18, 1983 of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS.

Discussion on this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N.Y. 10017, and will be accepted until two months after final publication of the paper itself in the JOURNAL OF APPLIED MECHANICS. Manuscript received by ASME Applied Mechanics Division, September, 1982; final revision, April, 1983. Paper No. 83-WA/APM-20.

Copies will be available until July, 1984.

$$D(W,_{XXXX}+W,_{YYYY}+2W,_{XXYY})+\rho\ddot{W}$$
$$=F,_{YY}(W+W_0),_{XX}+F,_{XX}(W+W_0),_{YY}$$
$$-2F,_{XY}(W+W_0),_{XY} \tag{1}$$

$$[1/(\mathrm{E}h)](F,_{XXXX}+F,_{YYYY}+2F,_{XXYY})$$
$$=[(W+W_0),_{XY}]^2-(W_{0,XY})^2$$
$$-(W+W_0),_{XX}(W+W_0),_{YY}+W_{0,XX}W_{0,YY} \tag{2}$$

In the foregoing, ($\ddot{\ }$) denotes differentiation twice with respect to time t, D is the flexural rigidity, E is Young's modulus, h is the plate thickness, ρ is the plate mass per unit area, and X and Y are the two in-plane coordinates. Introducing the nondimensional quantities w, w_0, f, x, and y defined by (ν is Poisson's ratio and b is the distance between the two edges parallel to the axial x coordinate),

$$(w,w_0)=(W,W_0)/h$$
$$f=2cF/(\mathrm{E}h^3)$$
$$c=[3(1-\nu^2)]^{1/2}$$
$$(x,y)=(1/b)(X,Y) \tag{3}$$

the governing nonlinear differential equations can be written in nondimensional form,

$$w,_{xxxx}+w,_{yyyy}+2w,_{xxyy}+[4\rho c^2b^4/(\mathrm{E}h^3)]\ddot{w}$$
$$=(2c)[f,_{yy}(w+w_0),_{xx}+f,_{xx}(w+w_0),_{yy}-2f,_{xy}(w+w_0),_{xy}] \tag{4}$$

$$f,_{xxxx}+f,_{yyyy}+2f,_{xxyy}$$
$$=(2c)\Big\{[(w+w_0),_{xy}]^2-(w_{0,xy})^2$$
$$-(w+w_0),_{xx}(w+w_0),_{yy}+w_{0,xx}w_{0,yy}\Big\} \tag{5}$$

The boundary conditions are taken to be simple support of the form (a is the length of the plate in the x direction),

$$w(y=0 \quad \text{or} \quad 1)=0$$
$$w,_{yy}(y=0 \quad \text{or} \quad 1)=0$$
$$w(x=0 \quad \text{or} \quad a/b)=0$$
$$w,_{xx}(x=0 \quad \text{or} \quad a/b)=0 \tag{6}$$

Furthermore, there is no in-plane shear along all the edges and the in-plane displacements normal to the edges are constant. The no shear boundary conditions imply,

$$f,_{xy}(y=0 \quad \text{or} \quad 1)=0$$
$$f,_{xy}(x=0 \quad \text{or} \quad a/b)=0 \tag{7a}$$

The in-plane displacements normal to the edges are constant, resulting in uniform normal stresses along the four edges. That is,

$$f,_{xx}(y=0 \quad \text{or} \quad 1)=\sigma_y=-N_y/h$$
$$f,_{yy}(x=0 \quad \text{or} \quad a/b)=\sigma_x=-N_x/h \tag{7b}$$

where N_x and N_y are the applied compressive stress resultants (force/unit length).

3 Previbration State

The previbration state of static equilibrium of a biaxially compressed plate is,

$$f_p(x,y)=(\bar{\sigma}_x)(-y^2/2)+(\bar{\sigma}_y)(-x^2/2)+f^*(x,y)$$
$$w_p(x,y)=w^*(x,y) \tag{8}$$

In the foregoing, the nondimensional compressive loads in the x and y directions are defined to be

$$(\bar{\sigma}_x,\bar{\sigma}_y)=[2cb^2/(\mathrm{E}h^3)](-N_x,-N_y) \tag{9}$$

Substituting $w=w_p(x,y)$ and $f=f_p(x,y)$ into equations (4) and (5), the appropriate nonlinear previbration equilibrium and compatibility equations are,

$$w^*,_{xxxx}+w^*,_{yyyy}+2w^*,_{xxyy}+(2c\bar{\sigma}_x)(w^*+w_0),_{xx}$$
$$+(2c\bar{\sigma}_y)(w^*+w_0),_{yy}$$
$$=(2c)[f^*,_{yy}(w^*+w_0),_{xx}+f^*,_{xx}(w^*+w_0),_{yy}$$
$$-2f^*,_{xy}(w^*+w_0),_{xy}] \tag{10}$$

$$f^*,_{xxxx}+f^*,_{yyyy}+2f^*,_{xxyy}$$
$$=(2c)\Big\{[(w^*+w_0),_{xy}]^2-(w_{0,xy})^2$$
$$-(w^*+w_0),_{xx}(w^*+w_0),_{yy}+w_{0,xx}w_{0,yy}\Big\} \tag{11}$$

The geometric imperfection is taken to be of the form

$$w_0(x,y)=\mu\sin(J\pi x)\sin(k\pi y) \tag{12}$$

where μ is the amplitude of the imperfection normalized with respect to the plate thickness, $J=jb/a$ and both j and k are positive integers. To avoid a discretized numerical solution, the preceding nonlinear previbration differential equations are solved in the following manner. From a physical point of view, it is reasonable to expect that the shape of the previbration displacement due to the in-plane preload is very similar to that of the initial geometric imperfection. Therefore, it is assumed that

$$w^*(x,y)=c_w\sin(J\pi x)\sin(k\pi y) \tag{13}$$

the nonlinear compatibility equation (11) becomes,

$$f^*,_{xxxx}+f^*,_{yyyy}+2f^*,_{xxyy}=(\pi^4cJ^2k^2)[(c_w+\mu)^2-\mu^2]$$
$$[\cos(2J\pi x)+\cos(2k\pi y)] \tag{14}$$

Thus, the stress function $f^*(x,y)$ which satisfies the compatibility equation exactly is

$$f^*(x,y)=[(c_w+\mu)^2-\mu^2][A_1\cos(2J\pi x)+A_2\cos(2k\pi y)] \tag{15}$$

where the constants A_1 and A_2 are found to be,

$$A_1=ck^2/(16J^2) \quad \text{and} \quad A_2=cJ^2/(16k^2) \tag{16}$$

Substituting $w^*(x,y)$, $f^*(x,y)$, and $w_0(x,y)$ into the nonlinear previbration equilibrium equation (10) and applying the Galerkin procedure (multiplying both sides by $\sin(J\pi x)\sin(k\pi y)$ and integrating from $x=0$ to a/b and $y=0$ to 1), one obtains a cubic equation in $c_w+\mu$ of the form,

$$[(4cJ^2k^2)(A_1+A_2)](c_w+\mu)$$
$$+[(J^2+k^2)^2-(J^2\bar{\sigma}_x+k^2\bar{\sigma}_y)(2c/\pi^2)$$
$$-(\mu^2)(4cJ^2k^2)(A_1+A_2)](c_w+\mu)^3$$
$$-(\mu)(J^2+k^2)^2=0 \tag{17}$$

Thus, for a given value of the axial or biaxial preload $\bar{\sigma}_x$ and $\bar{\sigma}_y$ and given values of the imperfection wave numbers and amplitude, one can find $c_w+\mu$ and hence $f^*(x,y)$. As a check, it can be seen that $\bar{\sigma}_x=\bar{\sigma}_y=0$ implies $c_w=0$. Care should be taken to ensure that prior to vibration, the preload does not cause excessive plate deformation.

4 Vibration of Imperfect Plates With Biaxial Preload

Using the perturbation procedure, the previbration state is added to the perturbed dynamic state $w_B(x,y,t)$ and $f_B(x,y,t)$ and the resulting differential equations are linearized with respect to the dynamic state. Doing so, the appropriate dynamic equilibrium and compatibility equations become,

$$w_{B,xxxx} + w_{B,yyyy} + 2w_{B,xxyy} + (4\rho c^2 b^4/(Eh^3))\ddot{w}_B$$
$$= (2c)\Big\{ (-\bar{\sigma}_x + f^*{,}_{yy})(w_{B,xx}) + (-\bar{\sigma}_y + f^*{,}_{xx})(w_{B,yy})$$
$$+ (w^* + w_0)_{,xx} f_{B,yy} + (w^* + w_0)_{,yy} f_{B,xx}$$
$$- 2(w^* + w_0)_{,xy} f_{B,xy} \Big\} \tag{18}$$

$$f_{B,xxxx} + f_{B,yyyy} + 2f_{B,xxyy}$$
$$= (2c)\Big\{ (2)(w^* + w_0)_{,xy} w_{B,xy} - (w^* + w_0)_{,xx} w_{B,yy}$$
$$- (w^* + w_0)_{,yy} w_{B,xx} \Big\} \tag{19}$$

The vibration mode that satisfies the boundary conditions is assumed to be

$$w_B(x,y,t) = \xi[\sin(M\pi x)\ \sin(n\pi y)]\ \exp(i\omega t) \tag{20}$$

where $i = (-1)^{1/2}$, ξ is the amplitude of the vibration mode normalized with respect to the plate thickness, ω is the frequency, $M = mb/a$, and both m and n are positive integers. Thus, the compatibility equation becomes,

$$f_{B,xxxx} + f_{B,yyyy} + 2f_{B,xxyy}$$
$$= \xi(c\pi^4/2)(c_w + \mu)$$
$$\Big\{ -(Jn - kM)^2 \cos[(J-M)\pi x]\cos[(k-n)\pi y]$$
$$+ (Jn + kM)^2 \cos[(J-M)\pi x]\cos[(k+n)\pi y]$$
$$+ (Jn + kM)^2 \cos[(J+M)\pi x]\cos[(k-n)\pi y]$$
$$-(Jn - kM)^2 \cos[(J+M)\pi x]\cos[(k+n)\pi y]\Big\}\exp(i\omega t) \tag{21}$$

It follows that the solution for the stress function $f_B(x,y,t)$ which satisfies the compatibility equation exactly is,

$$f_B(x,y,t) = \Big\{ c_1 \cos[(J-M)\pi x]\cos[k-n)\pi y]$$
$$+ c_2 \cos[(J-M)\pi x]\cos[(k+n)\pi y]$$
$$+ c_3 \cos[(J+M)\pi x]\cos[(k-n)\pi y]$$
$$+ c_4 \cos[(J+M)\pi x]\cos[(k+n)\pi y]\Big\}$$
$$\Big\{\xi(c_w + \mu)\exp(i\omega t)\Big\} \tag{22}$$

where the constants c_1, c_2, c_3, and c_4 are defined to be,

$$c_1 = \begin{cases} 0 & \text{if } Jn - kM = 0 \\ -(c/2)(Jn-kM)^2/[(J-M)^2 + (k-n)^2]^2 & \text{otherwise} \end{cases}$$
$$c_2 = (c/2)(Jn+kM)^2/[(J-M)^2 + (k+n)^2]^2$$
$$c_3 = (c/2)(Jn+kM)^2/[(J+M)^2 + (k-n)^2]^2 \tag{23}$$
$$c_4 = -(c/2)(Jn-kM)^2/[(J+M)^2 + (k+n)^2]^2$$

It may be observed that the stress function $f_B(x,y,t)$ satisfies the no shear boundary conditions.

The dynamic equilibrium equation is satisfied approximately using the Galerkin procedure by multiplying both sides by $\sin(M\pi x)\sin(n\pi y)$ and then integrating from $x = 0$ to a/b and $y = 0$ to 1. Doing so, one obtains an explicit expression for the frequency and the in-plane compressive loads in the form,

$$\Omega^2 + (\bar{\sigma}_x M^2 + \bar{\sigma}_y n^2)[c/2\pi^2)]$$
$$= -(2c)[(c_w + \mu)^2 - \mu^2][2M^2k^2A_2H_0 + 2J^2n^2A_1I_0]$$
$$+ (1/4)(M^2 + n^2)^2$$
$$-(2c)(c_w + \mu)^2\Big\{[J^2(k-n)^2 + k^2(J-M)^2]c_1I_1H_1$$
$$+ [J^2(k+n)^2 + k^2(J-M)^2]c_2I_1H_2$$
$$+ [J^2(k-n)^2 + k^2(J+M)^2]c_3I_2H_1$$
$$+ [J^2(k+n)^2 + k^2(J+M)^2]c_4I_2H_2$$
$$-(2Jk)[(J-M)(k-n)c_1I_3H_3 + (J-M)(k+n)c_2I_3H_4$$
$$+ (J+M)(k-n)c_3I_4H_3 + (J+M)(k+n)c_4I_4H_4]\Big\} \tag{24}$$

where

$$\Omega^2 = \rho\omega^2c^2b^4/(Eh^3\pi^4) \quad \text{or} \quad \Omega = [\omega cb^2/\pi^2][\rho/(Eh^3)]^{1/2}$$

and the definite integrals I_0, I_1, I_2, I_3, I_4, H_0, H_1, H_2, H_3, and H_4 are defined in Appendix A. In the important special case of no preload $\bar{\sigma}_x = \bar{\sigma}_y = 0$ and the geometric imperfection wave numbers $j = k = 1$, the fundamental frequency corresponds to $m = n = 1$ and the preceding frequency expression can be simplified to become ($c_w = 0$, $c_1 = c_4 = 0$),

$$\Omega_f = (1/2)\Big\{[(b/a)^2 + 1]^2 + [(b/a)^4 + 1](\mu^2c^2/2)\Big\}^{1/2} \tag{25}$$

5 Results and Discussion

To assess the effect of geometric imperfections on the vibration frequencies of flat rectangular plates, example problems are chosen from simply supported square ($a/b = 1$) and rectangular ($a/b = 5$) plates with Poisson's ratio of 0.3. Four types of geometric imperfections

$$\begin{aligned} &(1)\ J = k = 1 \\ &(2)\ J = 2,\quad k = 1 \\ &(3)\ J = 1,\quad k = 2 \\ &(4)\ J = k = 2 \end{aligned} \tag{26}$$

are considered and the uniaxial ($\bar{\sigma}_y = 0$) and equal biaxial ($\bar{\sigma}_x = \bar{\sigma}_y$) cases are examined. Due to symmetry consideration, a sign change in the imperfection amplitude will not affect the

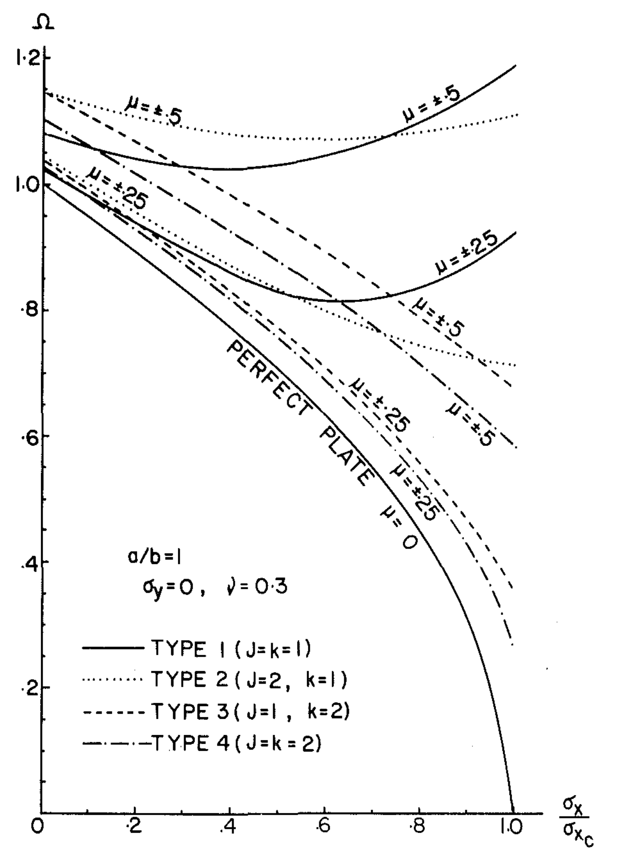

Fig. 1 Frequency versus uniaxial load interaction curves for simply supported square flat plates with imperfections

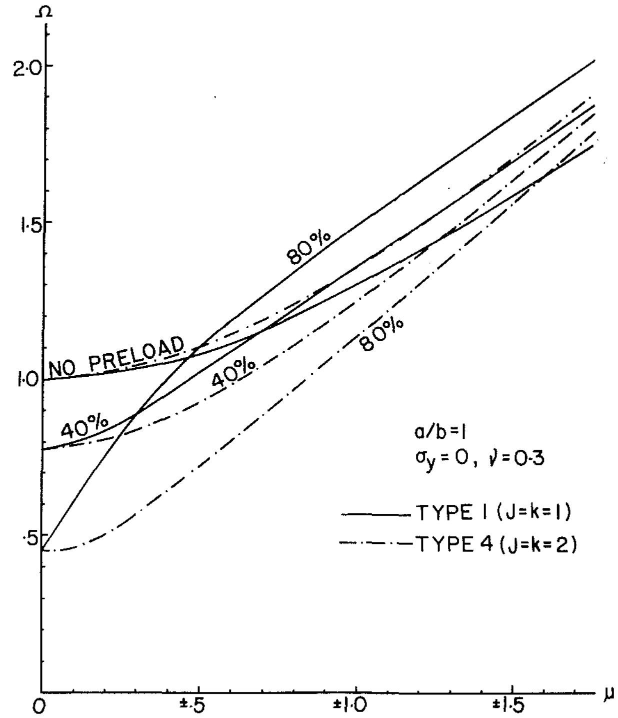

Fig. 2(*a*) Frequency versus imperfection amplitude for various values of uniaxial preload (imperfection types 1 and 4)

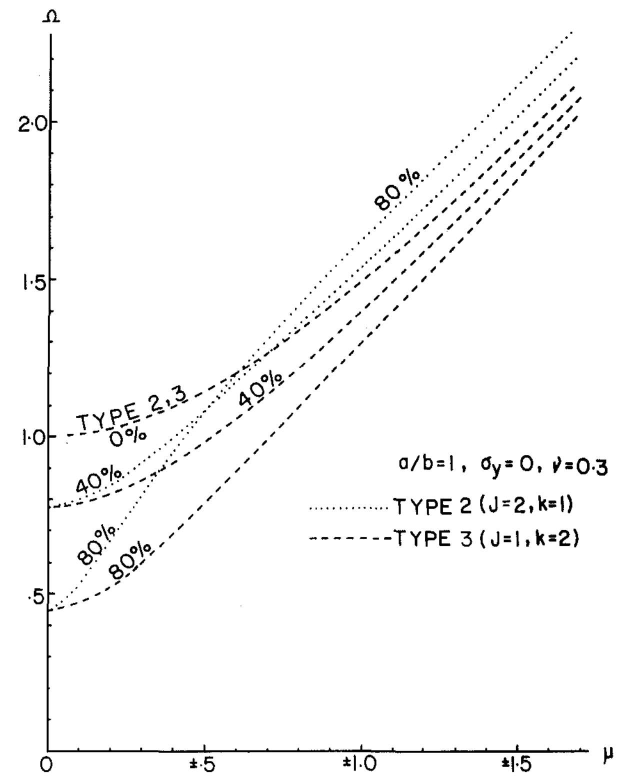

Fig. 2(*b*) Frequency versus imperfection amplitude for various values of uniaxial preload (imperfection types 2 and 3)

Fig. 2

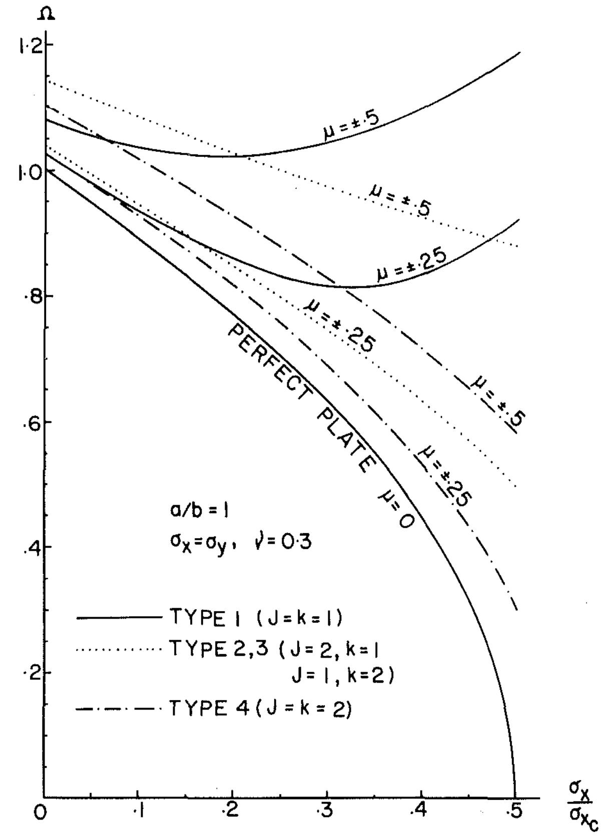

Fig. 3 Frequency versus equal biaxial load interaction curves for simply supported square flat plates with imperfections

problem. Unless otherwise specified, the optimum vibration discrete wave numbers M and n that render the frequency a minimum are $M=n=1$.

Figure 1 shows a graph of nondimensional frequency versus the uniaxial load ratio (σ_x/σ_{xc}), when σ_{xc} is the uniaxial critical buckling load which would exist with no imperfection ($\mu=0$). Curves are shown for values of imperfection amplitudes $\mu=0$, 0.25, and 0.5. It can be seen that the vibration frequency of an imperfect plate may be significantly higher than that for the perfect plate. Furthermore, the sensitivity to imperfections increases with the amount of uniaxial preload. It appears that frequencies are generally more sensitive to imperfection types 1 and 2 than to types 3 and 4. Whereas all frequency curves initially decrease with increasing initial in-plane stress, certain ones eventually increase (e.g., $J=k=1$) at larger in-plane initial stress values because of the increase of curvature as initial stress is increased.

Figures 2(*a*) and 2(*b*) show graphs of the vibration frequency versus imperfection ampltitude for $\sigma_x/\sigma_{xc}=0$, 40, and 80 percent. The vibration frequency may increase significantly due to the presence of geometric imperfection with amplitude of the order of only a fraction of the plate thickness. The observation that the sensitivity to imperfections increases with the amount of uniaxial preload is again supported by these figures.

A plot of the load interaction curves for frequency versus equal biaxial load for various values of the imperfection amplitude is shown in Fig. 3. The uniaxial buckling stress is again used as a normalizing factor. It can be observed that results for the uniaxial and equal biaxial cases are qualitatively similar. In fact, the biaxial loading curves for the imperfection type 1 ($J=K=1$) and type 4 ($J=k=2$) are identical to those of the uniaxial cases. Of course, under biaxial loading the curves for imperfection types 2 and 3 coincide due to symmetry. Figure 4 shows a graph of frequency versus imperfection amplitude for values of equal biaxial preload of 0 and 80 percent. These results are qualitatively similar to that presented in Figs. 2(*a*, *b*).

Figure 5 shows a plot of interaction curves for vibration

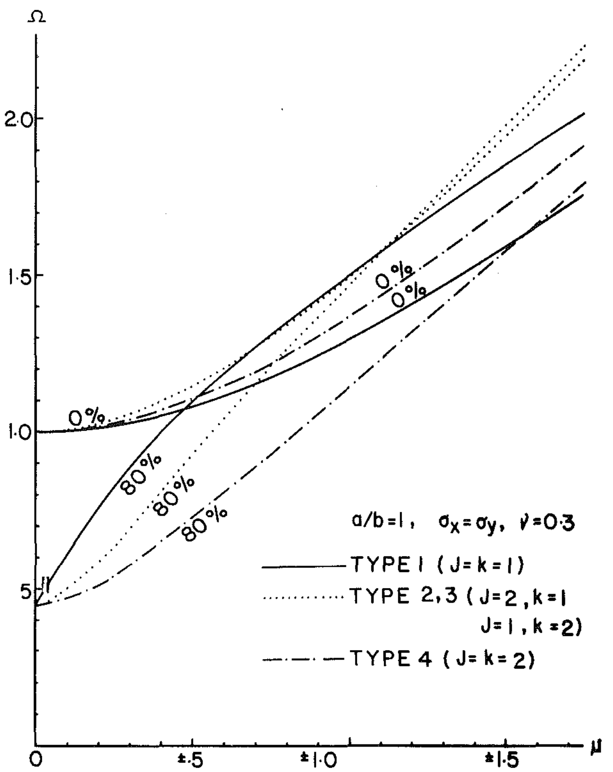

Fig. 4 Frequency versus imperfection amplitude for various values of equal biaxial preload

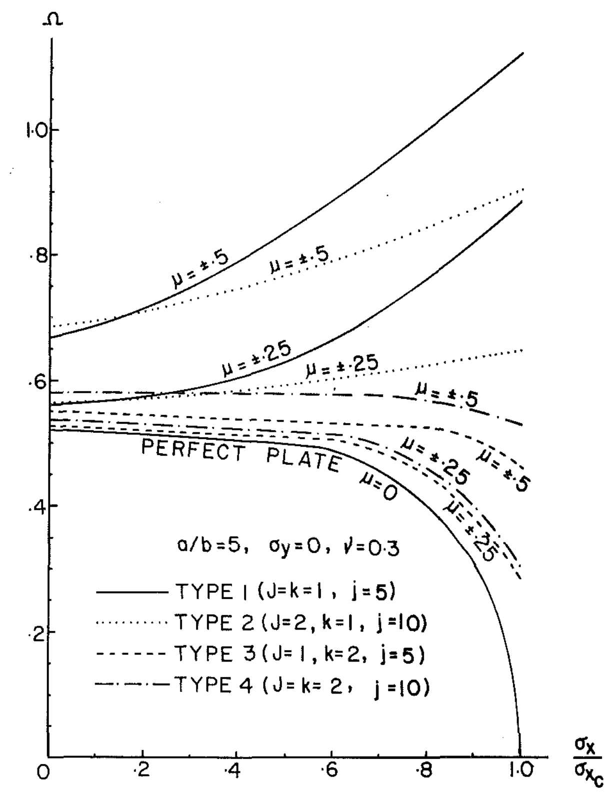

Fig. 5 Frequency versus uniaxial load interaction curves with various values of imperfection amplitudes for long rectangular plates ($a/b = 5$)

frequency versus uniaxial load with various values of imperfection amplitudes for long simply supported rectangular plates of aspect ratio $a/b = 5$ (so that $J = 1$ corresponds to $j = 5$ and $J = 2$ corresponds to $j = 10$). Again, it can be observed that imperfections may raise the frequencies significantly, especially in the case of large uniaxial preload. The vibration frequency is minimized for all possible discrete wave numbers m and n. The optimum value of n is always 1 whereas the optimum value of m varies from 1 (for sufficiently small values of uniaxial preload) to 5 (for sufficiently large preload). Furthermore, the presence of imperfections generally tends to reduce the optimum value of m. Since the optimum values of m change, the interaction curves should not be interpreted as perfectly smooth. They are the envelopes of separate curves obtained from each value of m. As the optimum value of m is 1 rather than 5 for sufficiently small amount of uniaxial preload, the nondimensional vibration frequencies for long plates ($a/b = 5$) are significantly smaller than those for square plates. It appears that the frequencies are more sensitive to imperfection types 1 and 2 than imperfection types 3 and 4.

Figures 6(a, b) show graphs of frequency versus imperfection amplitude with various values of uniaxial preload for long ($a/b = 5$) simply supported rectangular plates. The observations deduced from Fig. 5 are substantiated in these curves. The increase in frequency due to the amount of uniaxial preload is particularly more pronounced for imperfection types 1 and 2 than for imperfection types 3 and 4.

Figure 7 shows a graph of interaction curves for vibration frequency versus uniaxial load with values of imperfection amplitudes being 0, 0.25, 0.5, . . . , 1.5 for long simply supported rectangular plates of aspect ratio $a/b = 5$. The geometric imperfection assumes the shape of a half sine-wave in each of the two in-plane directions such that $J = 0.2$ (which corresponds to $j = 1$) and $k = 1$. For sufficiently large value of the imperfection amplitude, the optimum vibration frequency corresponds to $m = n = 1$. It is interesting to observe that the interaction curves tend to become more or less a horizontal straight line (i.e., independent of the in-plane preload) for the magnitude of the imperfection amplitudes greater than 0.75.

Finally, to provide an independent estimate of the effects of geometric imperfections on the vibration frequencies of simply supported flat plates, the frequencies of extremely shallow spherical shells of rectangular planform are studied. This permits direct comparison with the preceding imperfection type of analysis for type 1 imperfections ($J = k = 1$). The vibration frequency expression for spherical shells based on a solution of the linear Donnell type governing differential equation is (see Appendix B),

$$\Omega^2 + (2c/\pi^2)(M^2\bar{\sigma}_x + n^2\bar{\sigma}_y) = (M^2 + n^2)^2 + (Q^2/\pi^4) \qquad (27)$$

where $Q = 2cb^2/(Rh)$ and R is the shell radius.

The relevant results are tabulated in Table 1. It can be seen that significant increases in the vibration frequencies may occur for spherical shells with very small curvatures.

6 Concluding Remarks

The effects of various types of initial geometric imperfections on the small amplitude vibration frequencies of simply supported rectangular plates have been studied. It was found that the presence of small imperfections may significantly raise the frequencies, and the sensitivity to imperfections increases with the amount of uniaxial preload. The results for equal biaxial preloads are qualitatively similar to those for the case of uniaxial preload. The vibration frequencies for extremely shallow spherical shells of square planform supports qualitatively the increase in frequencies due to the presence of small amount of curvature. Further studies on the effects of geometric imperfections on the large amplitude vibrations of plates and shells were presented in separate papers [9–12].

7 Acknowledgments

The title problem was initiated and formulated by the first

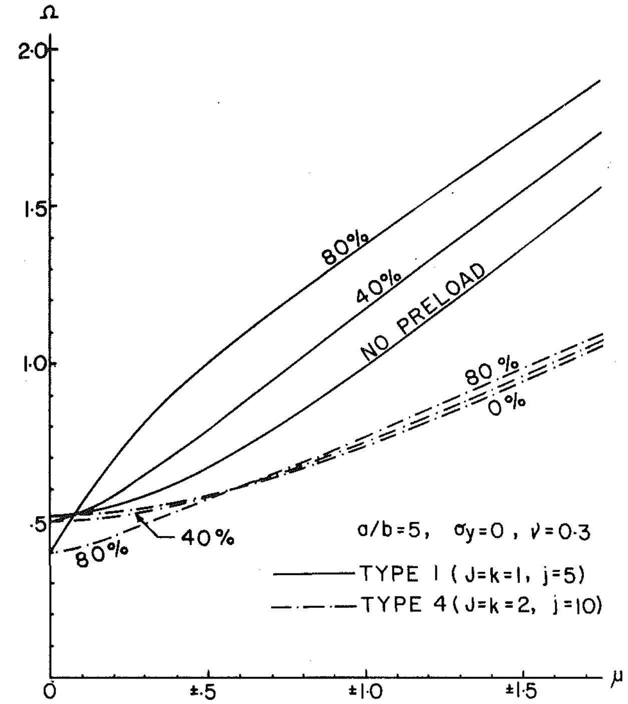

Fig. 6(a) Frequency versus imperfection amplitude with various values of uniaxial preload for long rectangular plates (imperfection types 1 and 4)

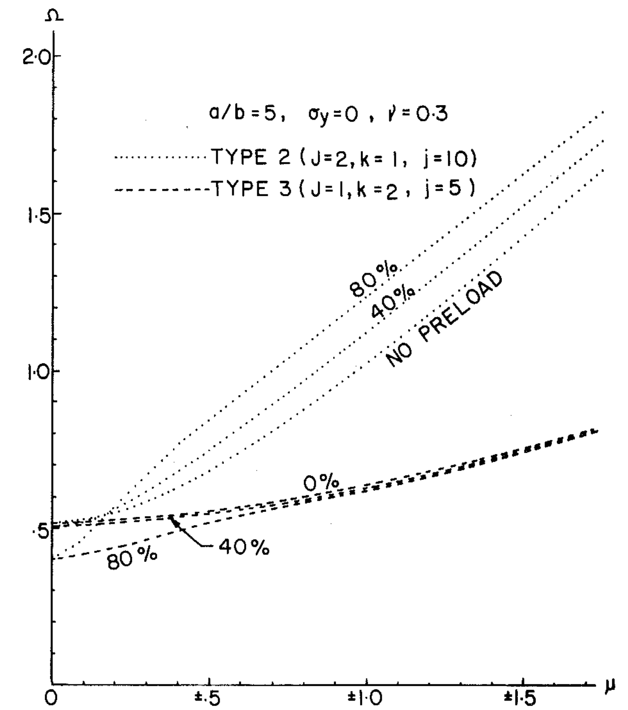

Fig. 6(b) Frequency versus imperfection amplitude with various values of uniaxial preload for long rectangular plates (imperfection types 2 and 3)

Fig. 6

author while he was a graduate student at the Department of Ocean Engineering, Massachusetts Institute of Technology. The second author wishes to thank Dr. I. H. Marshall of Paisley College, Scotland for an interesting discussion relevant to this topic in August, 1981.

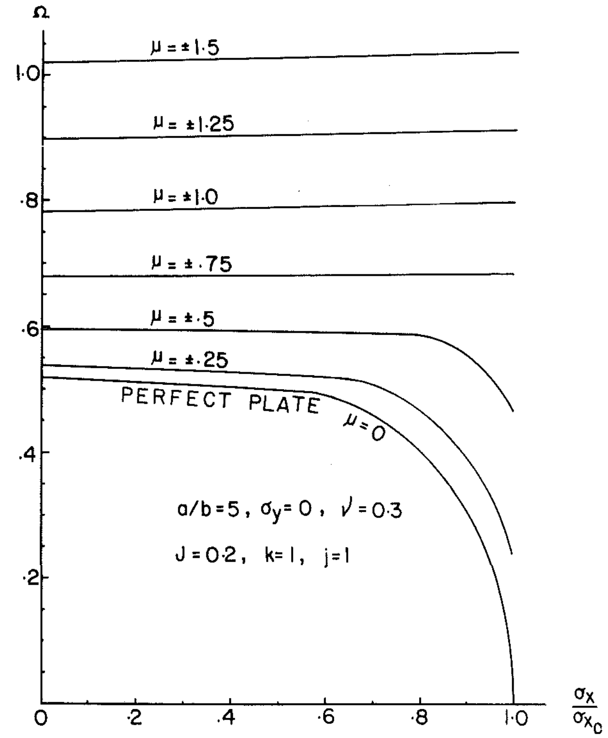

Fig. 7 Frequency versus uniaxial load interaction curves with various values of imperfection amplitudes for long rectangular plates ($a/b = 5$, $J = 0.2, k = 1, j = 1$)

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APPENDIX *A*

Definite Integrals

$$I_0 = (b/a)\int_0^{a/b} \sin^2(M\pi x) \quad \cos(2J\pi x)\ dx$$

$$= \begin{cases} -1/4 & \text{if} \quad M=J \\ 0 & \text{otherwise} \end{cases}$$

$$I_1 = (b/a)\int_0^{a/b} (\sin(J\pi x)\ \cos[(J-M)\,\pi x])\sin(M\pi x)\ dx$$

$$= \begin{cases} 1/2 & \text{if} \quad J=M \\ 1/4 & \text{otherwise} \end{cases}$$

$$I_2 = (b/a)\int_0^{a/b} (\sin(J\pi x)\ \cos[(J+M)\,\pi x])\sin(M\pi x)\ dx = -1/4$$

$$I_3 = (b/a)\int_0^{a/b} (\cos(J\pi x)\sin[(J-M)\,\pi x])\sin(M\pi x)\ dx$$

$$= \begin{cases} 0 & \text{if} \quad J=M \\ -1/4 & \text{otherwise} \end{cases}$$

$$I_4 = (b/a)\int_0^{a/b} (\cos(J\pi x)\sin[(J+M)\,\pi x])\sin(M\pi x)\ dx = 1/4$$

Furthermore, $H_0 = I_0$, $H_1 = I_1$, $H_2 = I_2$, $H_3 = I_3$, and $H_4 = I_4$ provided J and M are replaced by k and n, respectively, the factor b/a is deleted and integration is carried from $y=0$ to $y=1$.

APPENDIX *B*

Vibration of Extremely Shallow Spherical Shells of Rectangular Planform

The governing dynamic equilibrium and compatibility equations for a spherical shell of radius R can be obtained from those for the flat plate by adding the terms $(1/R)\,(F,_{XX}+F,_{YY})$ and $(-1/R)\,(W,_{XX}+W,_{YY})$ to the left-hand sides of equations (1) and (2), respectively. Keeping only the linear terms, the differential equations can be written in the nondimensional form,

$$w_{,xxxx} + w,_{yyyy} + 2w,_{xxyy} + [4\rho c^2 b^4/(\mathrm{E}h^3)]\ddot{w} \qquad (B1)$$

$$+ (2c)\,(\bar{\sigma}_x w,_{xx} + \bar{\sigma}_y w,_{yy}) + (Q)\,(f,_{xx} + f,_{yy}) = 0$$

$$(f,_{xxxx} + f,_{yyyy} + 2f,_{xxyy}) - (Q)\,(w,_{xx} + w,_{yy}) = 0 \qquad (B2)$$

where $Q = 2cb^2/(Rh)$. The vibrational mode is assumed to be,

$$w(x,y,t) = \xi_1[\sin(M\pi x)\sin(n\pi y)]\exp(i\omega t)$$

$$f(x,y,t) = \xi_2[\sin(M\pi x)\sin(n\pi y)]\exp(i\omega t) \qquad (B3)$$

so that the equilibrium and compatibility equations become,

$$[(M^2+n^2)^2 - \Omega^2 - (2c/\pi^2)(M^2\bar{\sigma}_x + n^2\bar{\sigma}_y)]\xi_1$$

$$-(Q/\pi^2)(M^2+n^2)\xi_2 = 0 \qquad (B4)$$

$$(M^2+n^2)\xi_2 + (Q/\pi^2)\xi_1 = 0 \qquad (B5)$$

Thus, the explicit expression for the vibration frequency squared is,

$$\Omega^2 + (2c/\pi^2)(M^2\bar{\sigma}_x + n^2\bar{\sigma}_y) = (M^2+n^2)^2 + (Q^2/\pi^4) \qquad (B6)$$

In the special case of $R/h=1000$, $R/b=2.5$, $\nu=0.3$, and $a/b=1$ one obtains $Q=528.727$ and

$$\omega_{\text{sphere}}/\omega_{\text{flat plate}} = 26.8043 \qquad (B7)$$

which agrees with the tabulated results obtained by Leissa and Kadi [15].

Assuming that the spherical shell is resting on a flat planform such that the maximum distance between the spherical shell middle surface and the flat reference plane is ϵ, an approximate comparison between the spherical shell and the imperfect flat plate can be made by equating the nondimensional imperfection amplitude μ to ϵ/h. Furthermore, from the geometry, one obtains

$$b/R = 2\cos^{-1}[1-(\epsilon/h)\,(h/R)] \qquad (B8)$$

Thus, by fixing $R/h=1000$ and $\nu=0.3$, the value of Q can be found based on the inputed values of ϵ/h. The computed frequency can then be normalized with respect to that for the flat plate ($\omega_{\text{sphere}}/\omega_{\text{flat plate}}$). The relevant results are presented in Table 1 for the case of no initial in-plane stress.

Table 1 Frequency of a spherical shell normalized with respect to that for the flat plate for $a/b=1$, $R/h=1000$, $\nu=0.3$ and $\Omega=[\omega cb^2/\pi^2][\rho/(\mathrm{E}h^3)]^{1/2}$

$\mu=\epsilon/h$	b/R	Q	Ω	$\omega/\omega_{\text{plate}}$
0.1	0.028285	2.6437	2.0179	1.0089
0.2	0.040001	5.2874	2.0705	1.0353
0.4	0.056570	10.575	2.2689	1.1345
0.5	0.063248	13.219	2.4071	1.2035
0.6	0.069286	15.863	2.5658	1.2829
0.8	0.080005	21.152	2.9314	1.4657
1.0	0.089450	26.441	3.3432	1.6716
1.5	0.10956	39.664	4.4890	2.2445
2.0	0.12651	52.890	5.7199	2.8599
3.0	0.15496	79.349	8.2848	4.1424
10.0	0.28308	264.81	26.905	13.452
20.0	0.40067	530.50	53.788	26.894