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EFFECTS OF UNI-DIRECTIONAL GEOMETRIC IMPERFECTIONS ON VIBRATIONS OF PRESSURIZED SHALLOW SPHERICAL SHELLS

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Abstract—This paper deals with the effects of initial geometric uni-directional imperfections on vibrations of a pressurized spherical shell or spherical cap. The analysis is based upon shallow shell theory. Frequency vs applied pressure interaction curves are plotted for various values of the imperfection amplitude. Imperfections are shown to have a severe effect in reducing the natural frequencies similar to that demonstrated in the buckling behavior of spherical shells.

INTRODUCTION

It is now well established that the discrepancies between the experimental and theoretical buckling loads of cylindrical or spherical shells are largely attributed to the presence of unavoidable initial geometric imperfections. Excellent reviews on the above subject can be found in articles written by Hutchinson and Koiter [1] and Kaplan [2]. The influence of imperfections on the vibrations of cylindrical shells has also been analyzed by Rosen and Singer [3, 4]. It appears from these papers that the effect of axisymmetric imperfection is to lower the natural frequencies whereas non-axisymmetric imperfections may either lower or raise them. Later, a more refined analysis including varying initial geometric imperfection shapes and vibration modes, considering single-valuedness of the circumferential displacements in a mixed formulation, with forced and free vibrations was presented by Watawala [5]. The vibrations of stringer-reinforced cylindrical shells having axisymmetric or asymmetric initial geometric imperfections with axial preload was analyzed by Singer and Prucz [6], using an equivalent orthotropic shell to represent the "smeared-out" stiffness. They found that small imperfections will change the natural frequencies of stiffened shells in the same directions as for isotropic shells, but to a smaller extent.

This study deals with the effects of uni-directional initial geometric imperfections on the vibrations of pressurized spherical shells. It appears that this subject has not been previously pursued for spherical shells in the literature. The analysis is based on a solution of the non-linear, Donnell shell equations in terms of a normal displacement and a stress function within the context of Koiter's "special theory" of elastic stability [7]. The pre-vibration state is taken to be a state of static equilibrium. It can be analyzed exactly since the non-linear terms in the Donnell equations automatically drop out due to the uni-directional character of the initial geometric imperfections. A two-term sinusoidal solution form is assumed for the vibration mode and the compatibility equation is satisfied exactly in terms of a three-term stress function. The equilibrium equation is then solved approximately using a Galerkin procedure. The result is a two by two homogeneous system of equations and the frequency is obtained by setting the determinant equal to zero. The minimum frequency is sought for all possible wave numbers. Since the compatibility equation is solved exactly and the equilibrium equation approximately, the present procedure is equivalent to the Ritz method so that the computed natural frequencies will be upper bounds.

The above analysis is simplified by assuming that the spherical shell is sufficiently shallow so that the wavelength of the vibration mode is small compared with the shell radius. This enables one to neglect all boundary conditions because of the periodicity requirements of the vibration mode [8–10].

GOVERNING DIFFERENTIAL EQUATIONS AND PRE-VIBRATION STATE

The governing differential equations consist of a dynamic equilibrium equation and a compatibility equation written in terms of a normal displacement W and a stress function F in the form [8],

$$(D)(W_{,XXXX} + W_{,YYYY} + 2W_{,XXYY}) + (1/R)(F_{,XX} + F_{,YY}) = F_{,XX}W_{,YY} + F_{,YY}(W + W_o)_{,XX} - 2F_{,XY}W_{,XY} - P - \rho\ddot{W} \quad (1)$$

$$[1/(Eh)](F_{,XXXX} + F_{,YYYY} + 2F_{,XXYY}) - (1/R)(W_{,XX} + W_{,YY}) = (W_{,XY})^2 - (W + W_o)_{,XX}W_{,YY} \quad (2)$$

where ($\ddot{}$) denotes differentiation of the variable twice with respect to time (t), ρ is the mass density, E is Young's modulus, D is the flexural rigidity, R is the radius, h is the shell thickness, P is the applied pressure (positive for external pressure), X and Y are the two in-plane coordinates and W_o is the initial geometric imperfection which is assumed to be a function only of the X coordinate. Introducing the non-dimensional quantities w, f, p, x and y defined by (ν is the Poisson's ratio),

$$w = W/h,\ w_o = W_o/h, \qquad f = 2cF/(Eh^3)$$
$$(x, y) = (q_o/R)(X, Y), \qquad q_o = (2cR/h)^{1/2}, \qquad c = [3(1 - \nu^2)]^{1/2}$$
$$p = cR^2P/(2Eh^2) \quad (3)$$

the governing non-linear differential equations become,

$$(w_{,xxxx} + w_{,yyyy} + 2w_{,xxyy}) + (f_{,xx} + f_{,yy}) + [\rho R^2/(Eh)]\ddot{w} + (2p/c) = (2c)\{f_{,xx}w_{,yy} + f_{,yy}(w + w_o)_{,xx} - 2f_{,xy}w_{,xy}\} \quad (4)$$

$$(f_{,xxxx} + f_{,yyyy} + 2f_{,xxyy}) - (w_{,xx} + w_{,yy}) = (2c)\{(w_{,xy})^2 - (w + w_o)_{,xx}w_{,yy}\}. \quad (5)$$

Assuming that the geometric imperfection takes the sinusoidal form,

$$w_o(x) = -\mu\cos(\alpha x) \quad (6)$$

the pre-vibration, static solution can be written in the form,

$$w_p = w^*(x) + c_o$$
$$f_p = c_1(x^2/2) + c_1(y^2/2) = f^*(x). \quad (7)$$

Substituting $w = w_p$ and $f = f_p$ into the dynamic equilibrium and compatibility equations (4) and (5), the non-linear terms automatically drop out, so that they become

$$w^*(x)_{,xxxx} + f^*(x)_{,xx} + 2c_1 = (2cc_1)[w^*(x) + w_o]_{,xx} - (2p/c)$$
$$f^*(x)_{,xxxx} - w^*(x)_{,xx} = 0. \quad (8)$$

Solving the above differential equations yields,

$$[w^*(x), f^*(x)] = (c_w, c_f)\cos(\alpha_x)$$
$$c_w = \frac{-2p\mu\alpha^2}{\alpha^4 - 2p\alpha^2 + 1} \quad (9)$$
$$c_f = \frac{2p\mu}{\alpha^4 - 2p\alpha^2 + 1}$$
$$c_1 = -p/c$$

and the arbitrary constant c_o may be set equal to zero.

VIBRATIONS OF PRESSURIZED SPHERICAL SHELLS

Using a perturbation procedure, the pre-vibration static equilibrium state w_p and f_p is added to the perturbed dynamic state $w_B(x, y, t)$ and $f_B(x, y, t)$ and the governing differential equations are then linearized in w_B and f_B. Thus, the appropriate equilibrium and compatibility equations are,

$$\begin{aligned}(w_{B,xxxx} + w_{B,yyyy} + 2w_{B,xxyy}) + (f_{B,xx} + f_{B,yy}) + [\rho R^2/(Eh)]\ddot{w}_B \\ + (2p)(w_{B,xx} + w_{B,yy}) \\ + (2c\alpha^2)\cos(\alpha x)\{c_f w_{B,yy} + (c_w - \mu)f_{B,yy}\} = 0\end{aligned} \tag{10}$$

$$f_{B,xxxx} + f_{B,yyyy} + 2f_{B,xxyy} = (2c\alpha^2)(c_w - \mu)\cos(\alpha x)w_{B,yy} + (w_{B,xx} + w_{B,yy}). \tag{11}$$

A two-term solution for $w_B(x, y, t)$ is assumed

$$w_B(x, y, t) = [d_o\cos(\alpha x/2) + d_1\cos(3\alpha x/2)]\cos(\gamma y)\exp(i\omega t) \tag{12}$$

where $i = \sqrt{-1}$ and ω is the frequency. The compatibility equation is satisfied exactly by letting $f_B(x, y, t)$ take the form,

$$\begin{aligned}f_B(x, y, t) = [(a_o d_o + a_2 d_1)\cos(\alpha x/2) + (a_1 d_o + a_3 d_1)\cos(3\alpha x/2) \\ + a_4 d_1\cos(5\alpha x/2)]\cos(\gamma y)\exp(i\omega t)\end{aligned} \tag{13}$$

where the constants a_0, a_1, a_2, a_3 and a_4 are found to be,

$$\begin{aligned}a_0 &= (-1/Q) - [(z)(c_w - \mu)/Q^2] \\ a_1 &= (-z)(c_w - \mu)/G^2 \\ a_2 &= (-z)(c_w - \mu)/Q^2 \\ a_3 &= -1/G \\ a_4 &= (-z)(c_w - \mu)/[(5\alpha/2)^2 + \gamma^2]^2\end{aligned} \tag{14}$$

and

$$\begin{aligned}Q &= (\alpha/2)^2 + \gamma^2 \\ G &= (3\alpha/2)^2 + \gamma^2 \\ z &= (c\alpha^2\gamma^2).\end{aligned} \tag{15}$$

The equilibrium equation is then satisfied approximately using the Galerkin procedure. Substituting $w_B(x, y, t)$ and $f_B(x, y, t)$ into the dynamic equilibrium equation, multiplying both sides by $\cos(\alpha x/2)\cos(\gamma y)$ and integrating (and repeating this with $\cos(3\alpha x/2)\cos(\gamma y)$) one obtains,

$$\begin{bmatrix} D_{11} - (\Omega^2/2) & D_{12} \\ D_{21} & D_{22} - (\Omega^2/2)\end{bmatrix}\begin{bmatrix} d_0 \\ d_1\end{bmatrix} = \begin{bmatrix} 0 \\ 0\end{bmatrix} \tag{16}$$

where, $\Omega^2/2 = \rho R^2\omega^2/(Eh)$

$$\begin{aligned}D_{11} &= Q^2 - a_0 Q - 2pQ - (z)\{c_f + (c_w - \mu)(a_0 + a_1)\} \\ D_{12} &= -a_2 Q - (z)\{c_f + (c_w - \mu)(a_2 + a_3)\} \\ D_{21} &= -a_1 G - (z)\{c_f + (c_w - \mu)a_0\} \\ D_{22} &= G^2 - a_3 G - 2pG - (z)(c_w - \mu)(a_2 + a_4).\end{aligned} \tag{17}$$

Setting the determinant of the above homogeneous system to zero, the frequency can be determined from,

$$\Omega^2 = (D_{11} + D_{22}) \pm \sqrt{(D_{11} + D_{22})^2 - (4)(D_{11}D_{22} - D_{12}D_{21})}. \tag{18}$$

The smallest frequency for all possible wave numbers is of interest. As a check on the above analysis, suppose $\alpha = 1$ and only a one-term solution for $w_B(x, y)$ is sought in the form,

$$w_B(x, y) = d_o \cos(x/2) \cos(\gamma y) \exp(i\omega t) \tag{19}$$

the nondimensional frequency expression becomes,

$$\Omega^2/2 = D_{11} = (Q^2 - 2pQ + 1) - (\mu)[z/(1-p)][(2/Q) + p] + (\mu)^2[z/(1-p)]^2[(1/Q^2) + (1/G^2)] \tag{20}$$

When Ω^2 is set equal to zero, this agrees with the upper bound buckling load obtained by Hutchinson [8].

RESULTS

In order to assess the influence of uni-directional geometric imperfections on the vibrations of pressurized spherical shells, the minimum frequency is computed based on a given value of the applied pressure for all possible wave numbers (the optimum wave number γ generally lies between 0 and 1). Assuming that Poisson's ratio is 0.3, the frequency vs the applied pressure (positive for external and negative for internal) interaction curves are plotted for the perfect system $\mu = 0$ and for the imperfect system $\mu = 0.1$ and 0.5.

Figure 1 shows frequency squared vs applied pressure interaction curves for $\alpha = 1$. It can be seen that for small values of the imperfection amplitude ($\mu = 0.1$) there is a significant (40 %) reduction of the buckling load at $\Omega^2 = 0$, whereas there is only a small (2 %) reduction of the natural frequency at $p = 0$. For larger values of the imperfection amplitude $\mu = 0.5$, the reduction of the natural frequency at zero load is comparable to the reduction in the buckling load at zero frequency. A plot of frequency squared vs imperfection amplitude for various values of the applied internal and external pressure (Fig. 2) substantiates the above observations. Finally, interaction curves for other values of the wavelengths of the geometric imperfections $\alpha = 2$ and $\alpha = 1/2$ show qualitatively similar trends (Figs 3 and 4).

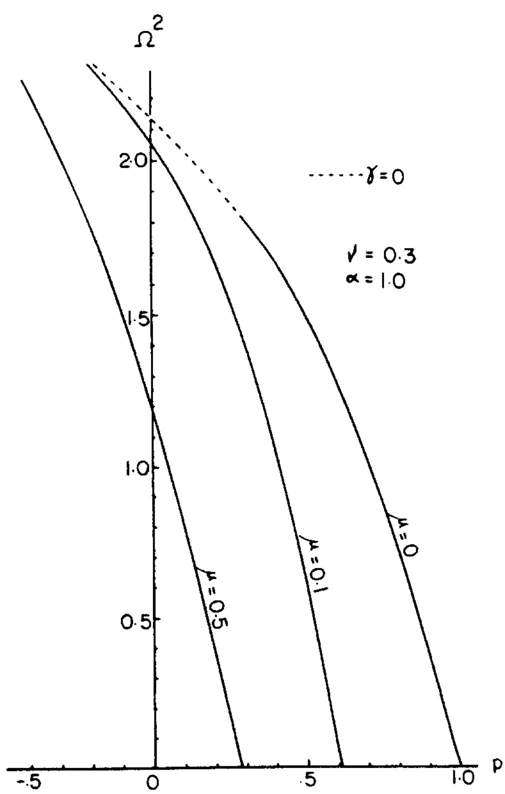

Fig. 1. Natural frequency squared vs applied pressure interaction curves for $\alpha = 1$.

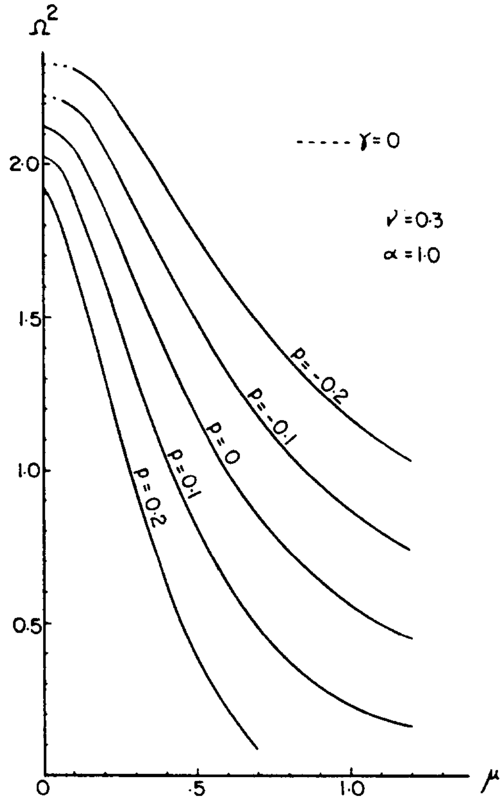

Fig. 2. Natural frequency squared vs imperfection amplitude for various values of external and internal pressure ($\alpha = 1$).

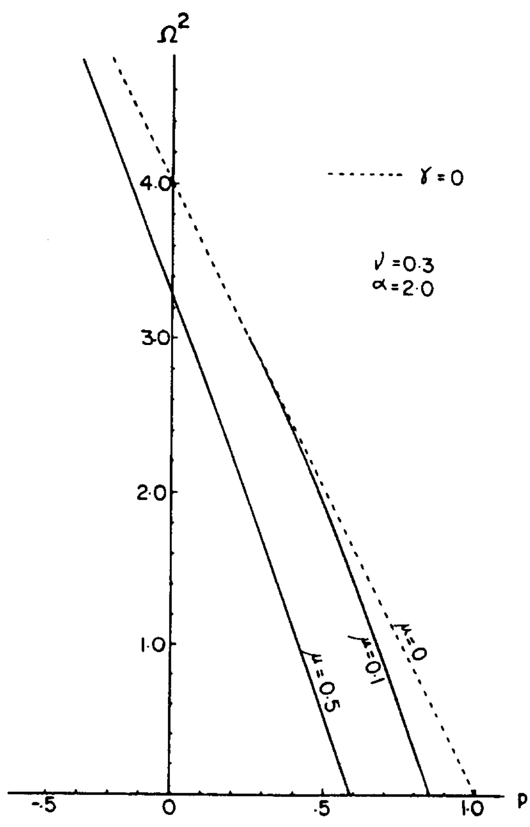

Fig. 3. Natural frequency squared vs applied pressure interaction curves for $\alpha = 2$.

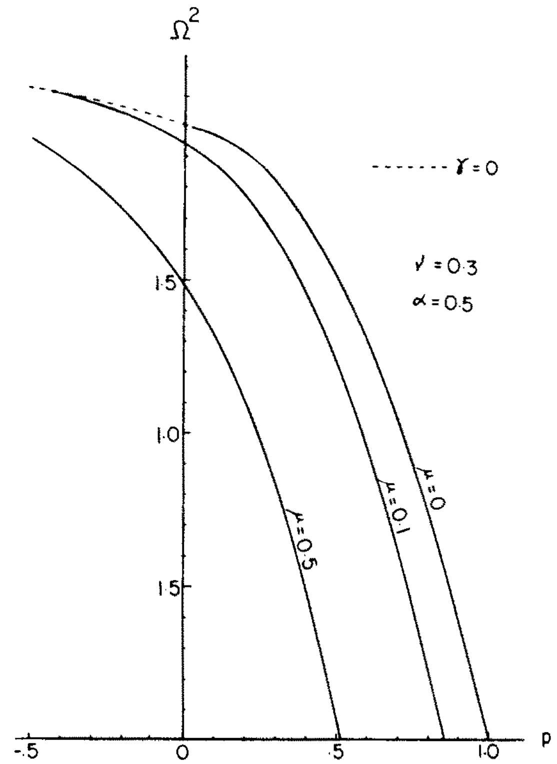

Fig. 4. Natural frequency squared vs applied pressure interaction curves for $\alpha = 1/2$.

CONCLUDING REMARKS

The effects of initial geometric uni-directional imperfections on the vibrations of pressurized spherical shells have been studied. It is found that imperfection plays a significant role in reducing the natural frequency of the system, even for zero pressure. It should be cautioned that the present analysis is valid only for vibration wavelengths which are small relative to the shell radius, so that the shallow shell theory which is used is applicable. Effects of geometric imperfections on large amplitude vibrations of simply supported and clamped circular plates [11], simply supported cylindrical panels with in-plane movable or immovable edges [12] and shallow spherical shells [13] can be found in the open literature.

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Résumé:

Cet article traite de l'effet d'imperfections géométriques initiales unidirectionnelles sur les vibrations de coques ou de calottes sphériques sous pression. L'analyse est basée sur la théorie des coques peu profondes. On dessine les courbes d'interaction de la fréquence en fonction de la pression appliquée pour différentes valeurs de l'amplitude de l'imperfection. On montre que les imperfections, en réduisant les fréquences naturelles, ont un effet important similaire à celui démontré dans le flambage des coques sphériques.

Zusammenfassung:

Diese Arbeit behandelt den Einfluss geometrischer, gerichteter Unvollkommenheiten auf die Schwingungen einer kugelfoermigen Schale oder eines kugelfoermigen Deckels unter Innendruck. Der Berechnung liegt die Theorie schwach gekruemmter Schalen zugrunde. Die gegenseitige Beinflussung von Frequenz und aufgebrachtem Druck wird fuer verschiedene Werte der Amplitude der Fehlstellen graphisch dargestellt. Es wird gezeigt, dass Fehlstellen die Eigenfrequenzen stark vermindern. Fuer das Knickverhalten kugelfoermiger Schalen wurde ein aehnlicher Einfluss demonstriert.