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LARGE AMPLITUDE AXISYMMETRIC VIBRATIONS OF GEOMETRICALLY IMPERFECT CIRCULAR PLATES

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This paper deals with the effects of geometric imperfections on the large amplitude vibrations of circular plates. It is found that geometric imperfections of the order of a fraction of the plate thickness may significantly raise the linear vibration frequencies. Furthermore, such imperfections may even change the inherent non-linear hard-spring character of the circular plates and cause them to exhibit soft-spring behavior. The effects of various boundary conditions are examined.

1. INTRODUCTION

Large amplitude vibrations of isotropic homogeneous flat plates is a topic which has been extensively investigated by many authors [1–5]. In particular, the effects of large amplitudes on axisymmetric vibration of circular plates were examined by Yamaki *et al.* [5, 6], Wah [7], Srinivasan [8], Huang and Al-Khattat [9] and Sridhar *et al.* [10]. Four different types of boundary conditions at the circular edge were examined by Yamaki [6] and it was found that the large amplitude vibrations of circular plates were always of the hardening type (frequencies increase with vibration amplitudes). This tendency is most pronounced for circular plates with simply supported, in-plane immovable edges.

On the other hand, the effects of geometric imperfections on linear and non-linear (large amplitude) vibrations of plates have received relatively little attention. It has been shown [11, 12] that geometric imperfections of the order of a fraction of the plate thickness may significantly raise the free vibration frequencies of simply supported rectangular plates. Moreover, they may even cause the plates to exhibit non-linear soft-spring behaviour instead of the inherent non-linear hard-spring characteristics of plates [12]. The large amplitude vibration behavior of thin square plates with very large geometric imperfections of the order of several times the plate thickness was investigated by Celep [13]. Large amplitude vibration of "clamped" circular plates with initial geometric imperfections and initial in-plane edge displacements was investigated by Yamaki *et al.* [5]. The results are primarily confined to only one value of the imperfection amplitude namely 10% of the plate thickness.

This paper deals with the effects of geometric imperfections on the linear and non-linear axisymmetric vibrations of circular plates. The vibration mode, the initial geometric imperfection and the forcing function are assumed to have the same spatial shape. In addition to the simply supported or clamped boundary conditions, two types of in-plane boundary conditions are considered: zero in-plane radial stress (in-plane movable) and zero in-plane radial displacement (in-plane immovable) at the circular edge. It is found that geometric imperfections of the order of a fraction of the plate thickness may significantly raise the free linear vibration frequencies. Imperfections (of magnitude $>10\%$ of the thickness) may even cause the circular plates to exhibit soft-spring behavior.

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The analysis is based on a solution of the dynamic analogue of the von Kármán non-linear equilibrium and compatibility equations, incorporating the possibility of the presence of an initial geometric imperfection. The complex modulus model [12, 14, 15] is used to incorporate the effects of structural damping on the linear forced vibration response.

2. ANALYSIS

The dynamic analogues of the von Kármán equilibrium and compatibility equations for axisymmetric vibrations of circular plates, written in terms of the out-of-plane displacement W and the Airy stress function F, are, respectively [4],

$$(1+\mathrm{i}\eta)(D)\bar{\nabla}^2(\bar{\nabla}^2 W)+\rho W_{,\bar{t}\bar{t}}-(1/\bar{r})[F_{,\bar{r}}(W+W_0)_{,\bar{r}}]_{,\bar{r}}=Q(\bar{r},\bar{t}), \tag{1}$$

$$\bar{\nabla}^2(\bar{\nabla}^2 F)=(1+\mathrm{i}\eta)(-Eh/\bar{r})[(W+W_0)_{,\bar{r}}W_{,\bar{r}\bar{r}}+W_{0,\bar{r}\bar{r}}W_{,\bar{r}}], \tag{2}$$

where the differential operator $\bar{\nabla}^2$ is defined to be

$$\bar{\nabla}^2(\quad)=(1/\bar{r})[\bar{r}(\quad)_{,\bar{r}}]_{,\bar{r}}. \tag{3}$$

Here $\bar{r}$ is the radial co-ordinate of the circular plate, E is Young's modulus, D is the flexural rigidity, h is the thickness, ρ is the mass of the plate per unit area, W_0 is the initial geometric imperfection, $\bar{t}$ is the time, $Q(\bar{r},\bar{t})$ is the forcing lateral pressure, $\mathrm{i}=(-1)^{1/2}$ and η is the loss factor associated with the complex modulus model for structural damping.

Upon introducing the non-dimensional quantities (ν is Poisson's ratio and a is the radius of the circular plate)

$$(w,w_0)=(W,W_0)/h,\qquad f=2cF/(Eh^3),\qquad r=\bar{r}/a,\qquad t=\bar{t}\omega_r,$$
$$c=[3(1-\nu^2)]^{1/2},\qquad q(r,t)=[a^4/(hD)]Q(\bar{r},\bar{t}), \tag{4}$$

where the reference frequency ω_r is defined to be

$$\omega_r=[D/(a^4\rho)]^{1/2}, \tag{5}$$

the non-linear dynamic equlibrium and compatibility equations become

$$(1+\mathrm{i}\eta)\nabla^2(\nabla^2 w)+w_{,tt}-(2c/r)[f_{,r}(w+w_0)_{,r}]_{,r}=q(r,t),$$
$$\nabla^2(\nabla^2 f)=(1+\mathrm{i}\eta)(-2c/r)[(w+w_0)_{,r}w_{,rr}+w_{0,rr}w_{,r}], \tag{6}$$

$$\nabla^2(\quad)=(1/r)[r(\quad)_{,r}]_{,r},\qquad \nabla^2(\nabla^2(\quad))=(1/r)\{r[\nabla^2(\quad)]_{,r}\}_{,r}. \tag{7}$$

The out-of-plane displacement is taken to be zero at the circular edge so that

$$w(r=1)=0. \tag{8}$$

Thus, the simply supported or clamped boundary conditions at $r=1$ are defined by specifying, respectively,

$$w_{,rr}(r=1)+\nu w_{,r}(r=1)=0,\quad\text{or}\quad w_{,r}(r=1)=0. \tag{9a,b}$$

Furthermore, in order to formulate the in-plane boundary conditions, it is desirable to express the membrane stress resultants N_r and N_t in terms of the in-plane radial displacement U and the out-of-plane displacement W in the form [16], presented in dimensional form for clarity purposes,

$$N_r=[(1+\mathrm{i}\eta)Eh/(1-\nu^2)][(\nu U/\bar{r})+U_{,\bar{r}}+(1/2)(W_{,\bar{r}})^2+W_{0,\bar{r}}W_{,\bar{r}}],$$
$$N_t=[(1+\mathrm{i}\eta)Eh/(1-\nu^2)][(U/\bar{r})+(\nu U_{,\bar{r}})+(\nu/2)(W_{,\bar{r}})^2+\nu W_{0,\bar{r}}W_{,\bar{r}}], \tag{10}$$

where $N_r = (1/\bar{r})F_{,\bar{r}}$ and $N_t = F_{,\bar{r}\bar{r}}$. Thus, U is related to the stress function by

$$F_{,\bar{r}\bar{r}} - (\nu/\bar{r})F_{,\bar{r}} = (1+\mathrm{i}\eta)(Eh)(U/\bar{r}). \tag{11}$$

Finally, the zero in-plane radial stress boundary condition at the circular edge is

$$f_{,r}(r=1)=0, \tag{12a}$$

while the zero in-plane radial displacement boundary condition is

$$f_{,rr}(r=1) - \nu f_{,r}(r=1) = 0. \tag{12b}$$

The out-of-plane displacement, the initial geometric imperfection and the forcing pressure are assumed to be of the same spatial shape:

$$[w(r,t),\, w_0(r),\, q(r,t)] = [w(t),\, \mu,\, q_1\cos(\omega t/\omega_r)](1+c_1r^2+c_2r^4). \tag{13}$$

For simply supported boundary conditions, the constants c_1 and c_2 are found to be

$$c_1 = -(6+2\nu)/(5+\nu) \quad \text{and} \quad c_2 = (1+\nu)/(5+\nu), \tag{14}$$

while for clamped boundary conditions they become

$$c_1 = -2 \quad \text{and} \quad c_2 = 1. \tag{15}$$

Substituting $w(r,t)$ and $w_0(r)$ into the non-linear compatibility equation, one obtains

$$[r(\nabla^2 f)_{,r}]_{,r} = [w(t)^2 + 2\mu w(t)](-8c)(c_1^2 r + 8c_1c_2r^3 + 12c_2^2r^5)(1+\mathrm{i}\eta). \tag{16}$$

Integrating, dividing and again integrating both sides with respect to r, one obtains

$$\nabla^2 f = [w(t)^2 + 2\mu w(t)](-2c)[c_1^2r^2 + 2c_1c_2r^4 + (4c_2^2r^6/3) + d_1](1+\mathrm{i}\eta). \tag{17}$$

A similar procedure yields

$$f_{,r} = [w(t)^2 + 2\mu w(t)](-2c)[(c_1^2r^3/4) + (c_1c_2r^5/3) + (c_2^2r^7/6) + (d_1r/2)](1+\mathrm{i}\eta). \tag{18}$$

The zero radial stress boundary condition at the circular edge implies that

$$d_1 = (-1)[(c_1^2/2) + (2c_1c_2/3) + (c_2^2/3)], \tag{19}$$

while the in-plane immovable boundary condition yields

$$d_1 = [-2/(1-\nu)]\{(3-\nu)(c_1^2/4) + (5-\nu)(c_1c_2/3) + (7-\nu)(c_2^2/6)\}. \tag{20}$$

Substituting $w(r,t)$, $w_0(r)$ and $f(r,t)$ into the dynamic equilibrium equation and applying the Galerkin procedure (multiplying both sides by $rw(r,t)$ and then integrating from $r=0$ to $r=1$), one obtains the well known Duffing's equation with an additional quadratic term: namely,

$$\begin{aligned}(1+\mathrm{i}\eta)64c_2I_0w(t) + G_0w(t)_{,tt} + (G_1)[w(t)^3 + 3\mu w(t)^2 + 2\mu^2 w(t)](1+\mathrm{i}\eta)\\ = q_1I_0\cos[(\omega/\omega_r)t].\end{aligned} \tag{21}$$

Here $(j=2,4,6,8)$

$$I_0 = \int_0^1 (r + c_1r^3 + c_2r^5)\,\mathrm{d}r = (1/2) + (c_1/4) + (c_2/6) \tag{22a}$$

$$I_j = \int_0^1 (r^j)(r + c_1r^3 + c_2r^5)\,\mathrm{d}r = [(1/(j+2)] + [c_1/(j+4)] + [c_2/(j+6)], \tag{22b}$$

$$G_0 = I_0 + c_1I_2 + c_2I_4, \tag{23a}$$

$$G_1 = (2c_1d_1I_0) + (8c_2d_1 + 2c_1^3)(I_2) + (10c_1^2c_2)(I_4) + (40c_1c_2^2/3)(2I_6) + (20c_2^3/3)(I_8). \tag{23b}$$

This Duffing-type differential equation can be written in standard form as

$$w(t)_{,tt}+(1+i\eta)\{(\varepsilon k)[w(t)^3+a_2 w(t)^2]+kw(t)\}=(q_1 I_0/G_0)\cos[(\omega/\omega_r)t], \quad (24)$$

$$k=(64c_2 I_0/G_0)+(4c^2)(2\mu^2)(G_1/G_0), \qquad \varepsilon k=(4c^2)(G_1/G_0), \qquad a_2=3\mu. \quad (25\text{–}27)$$

The solution of the linearized differential equation is

$$w(t)=A\cos[(\omega/\omega_r)t], \quad (28)$$

where the absolute value of the complex quantity A is $(z=(\omega/\omega_r)/\sqrt{k})$

$$|A|=\{q_1 I_0/(kG_0)\}/\{[1-z^2]^2+\eta^2\}^{1/2}. \quad (29)$$

The linear forced vibration response curves with structural damping have been tabulated and plotted in references [12] and [14].

The backbone curves for large amplitude vibrations of imperfect circular plates can be computed by solving the above non-linear differential equation with no damping and no forcing term. Using Linstedt's perturbation method, one can write the amplitude frequency relation in the form (see references [17, 18] and Appendix A)

$$\omega_{\text{non-linear}}/\omega_{\text{linear}}=z=1+r^*A^2+\cdots, \quad (30)$$

where the non-linearity parameter r^* is defined to be (the symbol r^* is used instead of r in order to avoid confusion with the radius of the circular plate)

$$r^*=(3\varepsilon/8)-(5a_2^2\varepsilon^2/12)=(3\varepsilon/8)[1-(10\mu^2\varepsilon)]. \quad (31)$$

Thus, at least for sufficiently small values of the vibration amplitude A, the non-linear hard-spring or soft-spring behaviour is indicated respectively, by positive or negative values of the non-linearity parameter r^* (the behavior tends to be more pronounced for larger magnitudes of r^*).

3. RESULTS AND DISCUSSION

Four types of boundary conditions at the circular edge ($r=1$) are considered in the present analysis:

(i) simply supported and $N_r(r=1)=0$; (ii) simply supported and $u(r=1)=0$;

(iii) clamped and $N_r(r=1)=0$; (iv) clamped and $u(r=1)=0$. (32)

With account taken of the presence of geometric imperfection, the present analysis yields Duffing-type differential equations ($\nu=0\cdot3$) for each of the above four types of boundary conditions, as follows:

(i) $$w(t)_{,tt}+(4c^2)\{[2\cdot24144+(1\cdot17762\mu^2)]w(t)+0\cdot588811[w(t)^3+3\mu w(t)^2]\}=1\cdot55920q_1\cos(\omega t/\omega_r); \quad (33)$$

(ii) $$w(t)_{,tt}+(4c^2)\{[2\cdot24144+(8\cdot29624\mu^2)]w(t)+4\cdot14812[w(t)^3+3\mu w(t)^2]\}=1\cdot55920q_1\cos(\omega t/\omega_r); \quad (34)$$

(iii) $$w(t)_{,tt}+(4c^2)\{[9\cdot76801+(2\cdot85714\mu^2)]w(t)+1\cdot42857[w(t)^3+3\mu w(t)^2]\}=1\cdot66666q_1\cos(\omega t/\omega_r); \quad (35)$$

(iv) $$w(t)_{,tt}+(4c^2)\{[9\cdot76801+(9\cdot20635\mu^2)]w(t)+4\cdot60317[w(t)^3+3\mu w(t)^2]\}=1\cdot66666q_1\cos(\omega t/\omega_r). \quad (36)$$

Thus it can be seen that the coefficients of the Duffing-type differential equation from the present analysis agree with those presented by Yamaki [6] for the special case of zero imperfection amplitude.

Figure 1 shows a graph of the linear vibration frequency versus imperfection amplitude for circular plates with Poisson's ratio 0·3. It can be seen that for a perfect circular plate ($|\mu| = 0$), the in-plane boundary condition has no effect on the linear vibration frequency. For imperfect plates, the increase in the linear frequency with increasing imperfection amplitude is much more pronounced for the $u(r = 1) = 0$ boundary condition than for the $N_r(r = 1) = 0$ condition. The linear frequencies agree with the results obtained by Yamaki [6] for perfect circular plates.

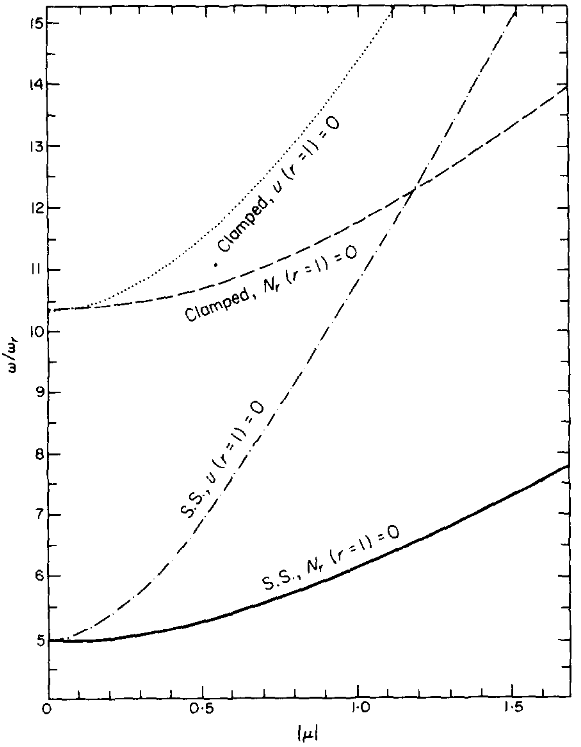

Figure 1. Linear vibration frequency versus amplitude of the geometric imperfection for circular plates with Poisson's ratio 0·3. $\omega_r = [D/(a^4\rho)]^{1/2}$.

Figure 2 shows a plot of the non-linearity parameter r^* versus the amplitude of the geometric imperfection for circular plates with Poisson's ratio of 0·3. It is apparent from equation (30) that the large amplitude vibration problem can be classified (at least for small values of the amplitude A) as hard-spring or soft-spring depending on whether r^* is positive or negative, respectively. The backbone curves based on the values of the non-linearity parameter for the perfect ($|\mu| = 0$) circular plates agree with the ones presented by Yamaki [6]. For each of the four types of boundary conditions, the non-linearity parameter decreases with increasing amplitudes of the geometric imperfection. Thus, the presence of imperfection of the order of a fraction of the plate thickness may cause the large amplitude vibrations of circular plates to exhibit soft-spring behavior (indicated by negative values of r^*). It should be noted that the non-linearity parameter eventually rises slightly (somewhat more rapidly for the simply supported $u(r = 1) = 0$ boundary condition) for large values of the amplitude of the geometric imperfection.

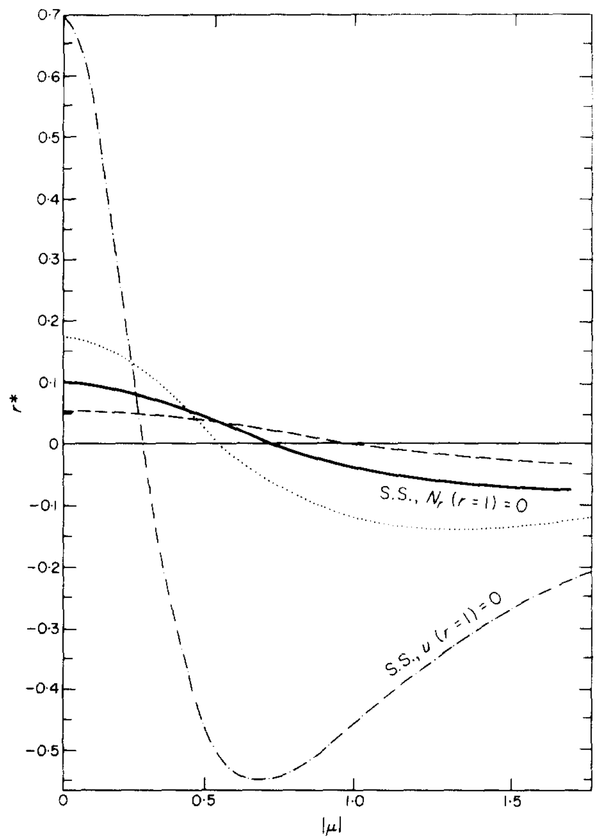

Figure 2. Non-linearity parameter versus amplitude of the geometric imperfection for circular plates with Poisson's ratio 0·3. $r^*>0$, Hard-spring; $r^*<0$, soft-spring. – – –, Clamped, $N_r(r=1)=0$; · · ·, clamped, $u(r=1)=0$.

Finally, the soft-spring behavior of shallow clamped spherical caps [19] supports qualitatively the soft-spring findings of the present paper.

4. CONCLUSION

The effects of geometric imperfections on large amplitude vibrations of circular plàtes have been investigated. It is found that the presence of geometric imperfections of the order of a fraction of the plate thickness may (1) significantly raise the linear free vibration frequencies and (2) change the inherent non-linear hard-spring character of the circular plates to soft-spring behavior. The influence of simply supported or clamped boundary conditions along with the zero radial stress or zero radial displacement boundary conditions are considered.

Extension of the present analysis to orthotropic circular plates [20], antisymmetric angle and cross-ply composite rectangular plates, and laminated cylindrical shells is in progress, and will be reported in due course.

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APPENDIX A: LINSTEDT'S PERTURBATION SOLUTION TO DUFFING'S EQUATION WITH A QUADRATIC TERM

The second order non-linear ordinary differential equation under consideration is

$$w(t)_{,tt}+kw(t)+(\varepsilon k)[a_2w(t)^2+w(t)^3]=0. \tag{A1}$$

By employing a change of variable, $\tau=\Omega t$, equation (A1) can be written in the form

$$\Omega^2w(\tau)_{,\tau\tau}+kw(\tau)=(-\varepsilon k)[a_2w(\tau)^2+w(\tau)^3]. \tag{A2}$$

The periodic solution $w(\tau)$ and the associated frequency (the period is $2\pi/\Omega$) are assumed to be of the forms

$$w(\tau)=w_0(\tau)+\varepsilon w_1(\tau)+\varepsilon^2w_2(\tau)+\cdots,\qquad \Omega=\Omega_0+\varepsilon\Omega_1+\varepsilon^2\Omega_2+\cdots. \tag{A3, A4}$$

Furthermore, the non-linear terms in equation A2 are also expanded in a power series in ε (see, for example, reference [18]). Equating terms involving ε^0, ε and ε^2 and dividing through by k (defined to be Ω_0^2), one obtains

$$w_0(\tau)_{,\tau\tau}+w_0(\tau)=0, \tag{A5}$$

$$w_1(\tau)_{,\tau\tau}+w_1(\tau)=-w_0(\tau)^3-a_2w_0(\tau)^2-2(\Omega_1/\Omega_0)w_0(\tau)_{,\tau\tau}, \tag{A6}$$

$$w_2(\tau)_{,\tau\tau}+w_2(\tau)=[-3w_0(\tau)^2-2a_2w_0(\tau)]w_1(\tau) \\ -[2(\Omega_2/\Omega_0)+(\Omega_1/\Omega_0)^2]w_0(\tau)_{,\tau\tau}-2(\Omega_1/\Omega_0)w_1(\tau)_{,\tau\tau},\cdots, \tag{A7}$$

and the initial conditions are

$$w_{0,\tau}(\tau=0)=0,\qquad w_{1,\tau}(\tau=0)=0,\qquad w_{2,\tau}(\tau=0)=0,\ldots. \tag{A8}$$

The solution of equation (A5) is

$$w_0(\tau)=A\cos(\tau). \tag{A9}$$

Substituting $w_0(\tau)$ into equation (A6), one obtains

$$w_1(\tau)_{,\tau\tau}+w_1(\tau)=-(a_2A^2/2)[1+\cos(2\tau)] \\ +[2(\Omega_1/\Omega_0)A-(3A^3/4)]\cos(\tau)-(A^3/4)\cos(3\tau). \tag{A10}$$

In order to avoid secular terms, the coefficient of $\cos(\tau)$ is set to zero so that

$$\Omega_1=(3/8)\Omega_0A^2. \tag{A11}$$

Accordingly, the solution to the differential equation for $w_1(\tau)$ is

$$w_1(\tau)=-(a_2A^2/2)+(a_2A^2/6)\cos(2\tau)+(A^3/32)\cos(3\tau). \tag{A12}$$

Finally, substituting the known forms for $w_0(\tau)$, $w_1(\tau)$ and Ω_1 into equation A6, one obtains

$$w_2(\tau)_{,\tau\tau}+w_2(\tau)=[(2A)(\Omega_2/\Omega_0)+(15A^5/128)+(5/6)a_2^2A^3]\cos(\tau) \\ +[(4/3)(\Omega_1/\Omega_0)(a_2A^2)-(a_2/32)A^4]\cos(2\tau) \\ +[(21A^5/128)-(a_2^2A^3/6)]\cos(3\tau) \\ -(a_2A^4/32)\cos(4\tau)-(3A^5/128)\cos(5\tau). \tag{A13}$$

Again, setting the coefficient of $\cos(\tau)$ to zero yields

$$\Omega_2/\Omega_0=-[(15A^4/256)+(5/12)a_2^2A^2]. \tag{A14}$$

Furthermore, the solution to the differential equation for $w_2(\tau)$ is

$$w_2(\tau)=(-1/3)[(4/3)(\Omega_1/\Omega_0)(a_2A^2)-(a_2/32)A^4]\cos(2\tau)-(1/8)[(21A^5/128) \\ -(a_2^2A^3/6)]\cos(3\tau)+(1/15)(a_2A^4/32)\cos(4\tau)+(A^5/1024)\cos(5\tau). \tag{A15}$$

The solution is obtained by assembling $w_0(\tau)$, $w_1(\tau)$ and $w_2(\tau)$ in equation A3 and replacing τ by $\Omega t+\phi$. Thus, for a given set of initial conditions $w(t=0)$ and $w_{,t}(t=0$, the value of the amplitude A and the phase angle ϕ can be found. Finally, substituting Ω_1 and Ω_2 into equation (A4) shows that the ratio of the non-linear frequency to the linear frequency is related to the vibration amplitude by

$$\Omega/\Omega_0=1+(\varepsilon)(3A^2/8)-(\varepsilon^2)[(15A^4/256)+(5a_2^2A^2/12)]. \tag{A16}$$

Re-arranging terms yields

$$\Omega/\Omega_0=1+(A^2)[(3\varepsilon/8)-(5a_2^2\varepsilon^2/12)]-(A^4)(15\varepsilon^2/256). \tag{A17}$$