Effects of Mode Interaction on Collapse of Short, Imperfect, Thin-Walled Columns

David Hui[1]
Department of Ocean Engineering,
Massachusetts Institute of Technology,
Cambridge, Mass. 02139.
Mem. ASME

The present paper deals with the design of beneficial geometric imperfections of short, thin-walled columns in order to maximize their energy absorption. The investigation was motivated by the experimental finding that under axial compressive load, the symmetric mode (which has a higher buckling load than the antisymmetric mode) actually has a much higher energy absorption than the antisymmetric mode as measured by the area under the curve of applied load versus end-shortening curve. Thus, an attempt is made to introduce imperfections in the beneficial symmetric mode so that the mode shapes of extremely large deflection in plastic collapse will also be of the symmetric type. The two-mode stability problem is studied using Koiter's theory of elastic stability.

1 Introduction

Crashworthiness of thin-walled structures has become one of the most important and challenging applied mechanics problems due to the fact that extremely large deformation is a complex mathematical problem. Moreover, it has found increasingly widespread applications in aerospace, mechanical, and ocean engineering structures. One of the most commonly encountered structural problems is the crushing of short, thin-walled columns under axial load. The dominant modes are the symmetric and the antisymmetric buckling modes of angled columns (see Fig. 1 and references [1-4]. The influence of initial geometric imperfections on the initial postbuckling behavior of such structures and their effects on the crushing of thin-walled structures are not well established. The present study is motivated by experimental investigations [1] which found that the symmetric mode of an angle-column has a much higher energy absorption than the antisymmetric mode as measured by the area under the load versus end-shortening curve for extremely large deflection (of the order of the length of the structure). An attempt is made to introduce the geometric imperfections in the "beneficial" symmetric mode so that the structure will follow the equilibrium paths of this mode in the initial postbuckling regime as well as in the regime of large deflection and crushing of the structure.

Previous investigations on the interaction between the Euler mode (one axial half wave) and the local mode (at least several axial half waves) of thin-walled columns was investigated by Van der Neut [5, 6], Koiter and Kuiken [7], and Byskov [8]. Related studies on the postbuckling behavior of thin-walled columns were also studied by Graves Smith [9, 10], and Grimaldi and Pignataro [11]. However, the mode interactions of short, angle-shaped, thin-walled columns where the two competing modes have the same axial wavelength (Euler modes) have not been investigated. Thus, it is of interest to study the initial postbuckling behavior of geometrically imperfect columns. In contrast to all the foregoing studies, the present ultimate goal is not to design a structure with high buckling load and low imperfection-sensitivity; rather, the objective is to design a structure with high energy absorption. Thus, an attempt is made to introduce a "beneficial" geometric imperfection of specified magnitude in the shape of the "symmetric mode" so that the structure will follow the equilibrium path of this mode in the initial postbuckling as well as in the very large deflection regimes, leading to the plastic collapse of the structure. Early experimental results on these angled structures are reported by Bridget, Jerome, and Vossler [12].

The present paper aims to examine the effects of geometric imperfections on the initial postbuckling behavior of short, angle-shaped, thin-walled columns under longitudinal compression. The loaded edges are assumed to be simply supported and one of the two longitudinal edges is free. The remaining longitudinal edge may be either simply supported (antisymmetric mode) or clamped (symmetric mode). Koiter's theory of multimode postbuckling [13, 14] is used to study the interaction between these two modes. The analysis is based on a solution of the nonlinear von Karman differential equations for plates valid for moderately large deflections. To compute the postbuckling coefficients, it is necessary to solve the eigenvalue problem (for the buckling loads and buckling modes of these two modes) and three uncoupled linear boundary-value problems for the second-order fields. Based

[1] Present affiliation: Assistant Professor, Department of Engineering Mechanics, The Ohio State University, Columbus, Ohio 43210.

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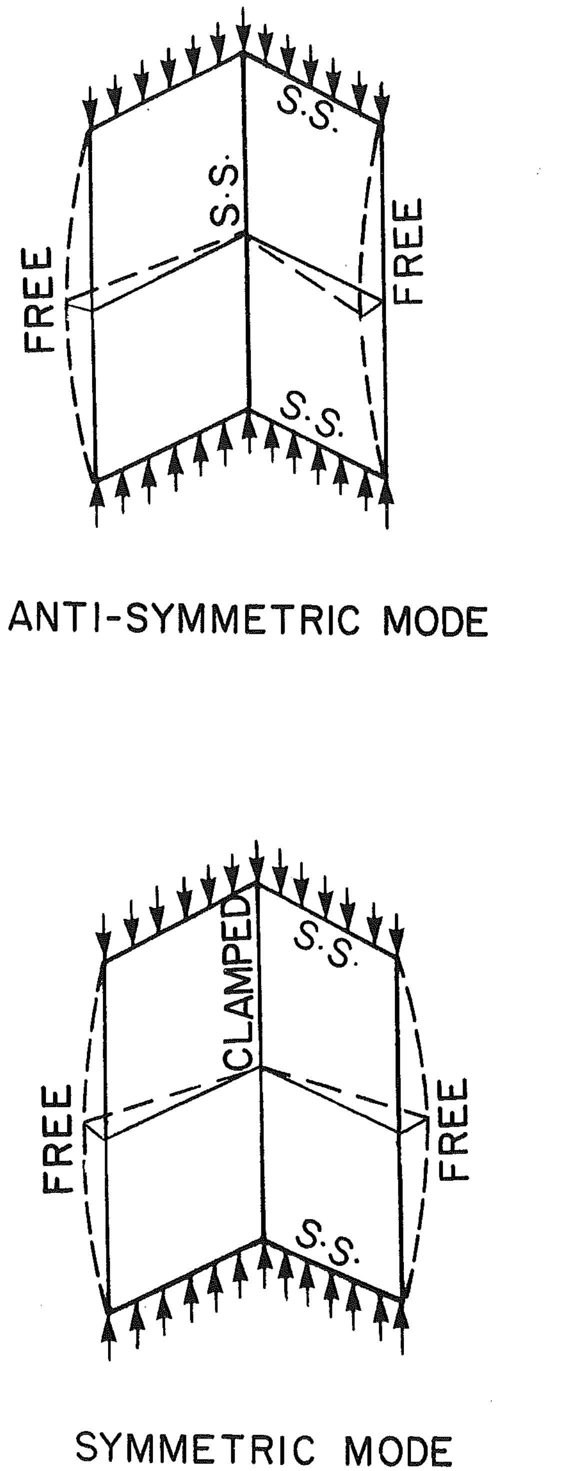

Fig. 1 Antisymmetric (simply supported at $Y=0$) and symmetric (clamped at $Y=0$) buckling modes of angled columns

on these postbuckling coefficients, the appropriate equilibrium paths of imperfect columns are plotted for Poisson's ratio of 0.3 and aspect ratio between 0.5 and 2.0. It is found that it is possible to design "beneficial" geometric imperfections so that the structure will follow the equilibrium paths of the symmetric mode in the large deflection regime.

2 Governing Differential Equations

The von Karman nonlinear equilibrium and compatibility equations for plates written in terms of an out-of-plane displacement W and a stress function F, valid for moderately large deflection, are

$$(D)\,(W,_{XXXX}+W,_{YYYY}+2W,_{XXYY})=F,_{YY}W,_{XX}+F,_{XX}W,_{YY}-2F,_{XY}W,_{XY} \quad (1)$$

$$[1/(Eh)](F,_{XXXX}+F,_{YYYY}+2F,_{XXYY})=(W,_{XY})^2-W,_{XX}W,_{YY} \quad (2)$$

where D is the flexural rigidity, E is Young's modulus, h is the plate thickness, and X and Y are the longitudinal and transverse in-plane coordinates. Introducing the nondimensional quantities (B is the width of the plate, $c=[3(1-\nu^2)]^{1/2}$ and ν is Poisson's ratio)

$$w=W/h,\quad f=2cF/(Eh^3),\quad (x,y)=(X,Y)(1/B) \quad (3)$$

the governing differential equations become

$$w,_{xxxx}+w,_{yyyy}+2w,_{xxyy}=(2c)\,[f,_{yy}w,_{xx}+f,_{xx}w,_{yy}-2f,_{xy}w,_{xy}] \quad (4)$$

$$f,_{xxxx}+f,_{yyyy}+2f,_{xxyy}=(2c)\,[(w,_{xy})^2-w,_{xx}w,_{yy}] \quad (5)$$

The plate is simply supported at the two loaded edges $x=0$ and $x=L/B$ where L is the length of the plate (see Fig. 1). Further, it is free at the edge $y=1$ so that

$$w,_{yy}(x,y=1)+\nu w,_{xx}(x,y=1)=0 \quad (6)$$

$$w,_{yyy}(x,y=1)+(2-\nu)\,w,_{xxy}(x,y=1)=0$$

Two types of boundary conditions at the intersecting edge $y=0$ are considered (Fig. 1): simply supported (i.e., antisymmetric mode) and clamped (i.e., symmetric mode). Finally, the stress resultants are related to the stress function by

$$N_x=F,_{YY},\quad N_y=F,_{XX},\quad N_{xy}=-F,_{XY} \quad (7)$$

and the in-plane boundary conditions will be examined subsequently.

Within the context of Koiter's theory of elastic stability [13], the out-of-plane displacement and the stress function for the two-mode stability problem are sought in the form (ξ_1 and ξ_2 are the amplitudes of the antisymmetric and symmetric modes, respectively)

$$w=0+(\xi_1w_1+\xi_2w_2)+(\xi_1{}^2w_{11}+\xi_2{}^2w_{22}+2\xi_1\xi_2w_{12})=w_c+w_{II} \quad (8)$$

$$f=f_0+(\xi_1f_1+\xi_2f_2)+(\xi_1{}^2f_{11}+\xi_2{}^2f_{22}+2\xi_1\xi_2f_{12})=f_0+f_c+f_{II}$$

It will be shown in subsequent sections that for the present problem

$$f_1=f_2=0 \quad \text{and} \quad w_{11}=w_{22}=w_{12}=0 \quad (9)$$

3 Classical Buckling Load

For a rectangular plate uniformly compressed in the longitudinal direction, buckling occurs with zero in-plane displacements. The differential equation for buckling is

$$w,_{xxxx}+w,_{yyyy}+2w,_{xxyy}+(\pi^2\sigma)\,w,_{xx}=0 \quad (10a)$$

where ($\bar{\sigma}$ is the axial stress per unit cross-sectional area, positive for compression, and f_0 is the prebuckling stress function)

$$\sigma=k(\text{Timoshenko})=\bar{\sigma}hB^2/(\pi^2D)=(-2c/\pi^2)f_{0,yy} \quad (10b)$$

Antisymmetric Mode (Simply Supported at $y=0$). The solution that satisfies the simply supported condition at the two loaded edges is

$$w_1(x,y)=w_1(y)\,\sin(M_1x) \quad (11)$$

where $M_1 = m_1 B/L$ and m_1 is the number of axial half waves. Thus, the ordinary homogeneous differential equation is

$$w_1(y)_{,yyyy} - 2M_1^2 w_1(y)_{,yy} + (M_1^4 - \pi^2 \sigma M_1^2) w_1(y) = 0 \quad (12)$$

The solution that satisfies the simply supported boundary condition at $y=0$ is [15]

$$w_1(y) = (\xi_1)[A_1 \sinh(c_1 y) + B_1 \sin(d_1 y)] \quad (13)$$

Substituting the buckling mode into the differential equation, the real positive quantities c_1 and d_1 are found to be functions of the applied load in the form

$$c_1 = [M_1^2 + \pi M_1 \sigma^{1/2}]^{1/2}, \quad d_1 = [-M_1^2 + \pi M_1 \sigma^{1/2}]^{1/2} \quad (14)$$

where $\sigma > (M_1^2/\pi^2) = (m_1 B/L)^2$. Further, by defining $w_1(y=1) = \xi_1$, the quantity ξ_1 can be considered as the amplitude of the antisymmetric buckling mode at the middle of the free edge normalized with respect to the plate thickness. Substituting $w_1(x,y)$ into the two boundary conditions at the free edge $y=1$, one obtains

$$\begin{aligned}&(c_1^2 - \nu M_1^2)[\sinh(c_1)]A_1 + (d_1^2 + \nu M_1^2)[-\sin(d_1)]B_1 = 0\\ &(c_1)[c_1^2 - (2-\nu)M_1^2][\cosh(c_1)]A_1\\ &\qquad + (d_1)[d_1^2 + (2-\nu)M_1^2][-\cos(d_1)]B_1 = 0\end{aligned} \quad (15)$$

Since the first principal minor is always positive, the condition for buckling is that the determinant should vanish, that is (using $c_1^2 - d_1^2 = 2M_1^2$),

$$(-d_1)(c_1^2 - \nu M_1^2)^2 \tanh(c_1) + (c_1)(d_1^2 + \nu M_1^2)^2 \tan(d_1) = 0 \quad (16)$$

Symmetric Mode (Clamped at $y=0$). Similarly, the symmetric mode that satisfies the boundary conditions at the loaded edges $x=0$ and $x=L/B$ and at the intersecting edge $y=0$ can be written in the separable form $w_2(x,y) = w_2(y) \sin(M_2 x)$ where $M_2 = m_2 \pi B/L$ and

$$\begin{aligned}w_2(y) = (\xi_2)\Big\{&A_2[\cos(d_2 y) - \cosh(c_2 y)]\\ &+ B_2[\sin(d_2 y) - (d_2/c_2)\sinh(c_2 y)]\Big\}\end{aligned} \quad (17)$$

In the foregoing, the positive real quantities c_2 and d_2 are obtained by replacing M_1 by M_2 in equations (14). By imposing $w_2(y=1) = \xi_2$, the quantity ξ_2 is the amplitude of the symmetric mode at the middle of the free edge normalized with respect to the plate thickness. The two boundary conditions at the free edge $y=1$ yield (using $c_2^2 - d_2^2 = 2M_2^2$)

$$\begin{aligned}&[-T\cos(d_2) - S \cosh(c_2)]A_2\\ &\quad -[T\sin(d_2) + (d_2/c_2)S \sinh(c_2)]B_2 = 0\\ &[Sd_2 \sin(d_2) - Tc_2 \sinh(c_2)]A_2\\ &\quad -[Sd_2 \cos(d_2) + Td_2 \cosh(c_2)]B_2 = 0\end{aligned} \quad (18)$$

where $S = c_2^2 - \nu M_2^2$ and $T = d_2^2 + \nu M_2^2$. The vanishing of the determinant yields (one should also check the vanishing of the first principal minor)

$$\begin{aligned}&2TS + (S^2 + T^2)\cos(d_2) \cosh(c_2)\\ &\quad = [(c_2^2 T^2 - d_2^2 S^2)/(c_2 d_2)]\sin(d_2) \sinh(c_2)\end{aligned} \quad (19)$$

4 Second-Order Fields

Within the context of Koiter's theory of elastic stability, the postbuckling behavior requires the computation of the second-order fields w_{II} and f_{II} in addition to the buckling mode w_c and f_c. Since $f_c = 0$, the differential equation for w_{II} is identical to that for w_c so that w_{II} is identically zero due to the orthogonality requirement between w_c and w_{II}. The differential equation for the stress function f_{II} is

$$\begin{aligned}&f_{II,xxxx} + f_{II,yyyy} + 2f_{II,xxyy}\\ &\qquad = (2c)[(w_{c,xy})^2 - w_{c,xx} w_{c,yy}]\end{aligned} \quad (20)$$

Substituting $w_c = w_1(x,y)$ into equation (20), the differential equation for f_{11} which corresponds to the antisymmetric mode is

$$\begin{aligned}&f_{11,xxxx} + f_{11,yyyy} + 2f_{11,xxyy}\\ &= (cM_1^2)\Big\{w_1(y)_{,y}{}^2 + w_1(y) w_1(y)_{,yy}\\ &\qquad + [w_1(y)_{,y}{}^2 - w_1(y) w_1(y)_{,yy}]\cos(2M_1 x)\Big\}\end{aligned} \quad (21)$$

Thus, the solution is of the separable form

$$f_{11}(x,y) = f_\alpha(y) + f_\beta(y)\cos(2M_1 x) \quad (22)$$

The ordinary differential equation for the $f_\alpha(y)$ problem is

$$f_\alpha(y)_{,yyyy} = (cM_1^2)[w_1(y)_{,y}{}^2 + w_1(y) w_1(y)_{,yy}] \quad (23)$$

while for the $f_\beta(y)$ problem, one obtains

$$\begin{aligned}&f_\beta(y)_{,yyyy} - (8M_1^2) f_\beta(y)_{,yy} + (16M_1^4) f_\beta(y)\\ &\qquad = (cM_1^2)[w_1(y)_{,y}{}^2 - w_1(y) w_1(y)_{,yy}]\end{aligned} \quad (24)$$

The in-plane boundary conditions at the free edge are $N_{xy}(y=1) = 0$ and $N_y(y=1) = 0$; that is,

$$f_{\beta,y}(y=1) = 0, \quad f_\beta(y=1) = 0 \quad (25)$$

Furthermore, there is no in-plane longitudinal shear at the intersecting edge $y=0$ due to symmetry; that is, $N_{xy}(y=0) = 0$, which implies

$$f_{\beta,y}(y=0) = 0 \quad (26)$$

From symmetry considerations, the in-plane displacement in the y direction, V, vanishes at the intersecting edge $y=0$. Since V is related to W and F by [16, 17]

$$\begin{aligned}(-Eh)(V_{,XX} + W_{,XX} W_{,Y}) &= (F_{,YY})_{,Y} - \nu(F_{,XX})_{,Y}\\ &\quad + (2+2\nu)(F_{,XY})_{,X}\end{aligned} \quad (27)$$

The foregoing boundary conditions imply

$$f_{\beta,yyy}(y=0) = 0, \quad f_{\alpha,yyy}(y=0) = 0 \quad (28)$$

At the two loaded edges $x=0$ and $x=L/B$, it can be seen that the no shear condition $N_{xy} = 0$ is satisfied exactly by the separable form of the stress function. Imposing the condition that the average second-order longitudinal stress is zero at the loaded edge, one obtains

$$\int_{y=0}^{1} \Big\{f_\alpha(y)_{,yy} + f_\beta(y)_{,yy}\Big\} dy = 0 \quad (29)$$

Furthermore, integrating both sides of the differential equation for $f_\alpha(y)$ twice, one obtains

$$f_\alpha(y)_{,yy} = (cM_1^2/2) w_1(y)^2 + c_\alpha \quad (30)$$

where the constant of integration c_α is found from equation (29) to be

$$c_\alpha = -cM_1^2 H_1/2 \quad \text{and} \quad H_1 = \int_0^1 w_1(y)^2\, dy \quad (31)$$

It will be shown that it is not necessary to compute $f_\alpha(y)$ nor $f_\alpha(y)_{,y}$ since the postbuckling coefficient depends only on $f_\alpha(y)_{,yy}$.

Closed-form solution for the $f_\beta(y)$ problem is sought as the sum of a particular solution and a homogeneous solution in the form

$$f_\beta(y) = P_\beta(y) + H_\beta(y) \quad (32)$$

where

$$\begin{aligned}P_\beta(y) = e_0 &+ e_1 \sin(d_1 y) \sinh(c_1 y)\\ &+ e_2 \cos(d_1 y) \cosh(c_1 y)\end{aligned}$$

$$H_\beta(y) = (s_1 + ys_2)\exp(2M_1 y) + (s_3 + ys_4)\exp(-2M_1 y) \quad (33)$$

Substituting the solution form into equation (24), the constants e_0, e_1, and e_2 which satisfy the differential equation exactly are

$$e_0 = (A_1^2 c_1^2 + B_1^2 d_1^2)c/(16M_1^2) \quad (34)$$

and e_1 and e_2 can be determined from

$$\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \quad (35)$$

In the foregoing,

$$(E_1, E_2) = (cM_1^2)(A_1 B_1)[d_1^2 - c_1^2, 2c_1 d_1]$$

$$E_{11} = E_{22} = (c_1^4 + d_1^4 - 6c_1^2 d_1^2) - (8M_1^2)(c_1^2 - d_1^2) + 16M_1^4 \quad (36)$$

$$E_{12} = (-1)E_{21} = (4c_1 d_1^3 - 4c_1^3 d_1) + 16c_1 d_1 M_1^2$$

Finally, the constants s_1, s_2, s_3, and s_4, which enable the total solution $P_\beta(y) + H_\beta(y)$ to satisfy the boundary conditions exactly, can be computed from

$$\begin{bmatrix} P+Q & -P+Q \\ (2M_1)(P-Q) & -(1+2M_1)P + (1-2M_1)Q \end{bmatrix} \begin{bmatrix} s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} Z_3 - (Z_5 + Z_6)P \\ Z_4 - P[2M_1 Z_5 + (1+2M)Z_6] \end{bmatrix} \quad (37)$$

where

$$P = \exp(2M_1), \quad Q = \exp(-2M_1)$$

and

$$Z_1 = -P_{\beta,y}(y=0), \quad Z_2 = -P_{\beta,yyy}(y=0)$$

$$Z_3 = -P_\beta(y=1), \quad Z_4 = -P_{\beta,y}(y=1)$$

$$Z_5 = [3Z_1/(4M_1)] - [Z_2/(16M_1^3)] \quad (38)$$

$$Z_6 = [Z_2/(8M_1^2)] - (Z_1/2)$$

The constants s_1 and s_2 can be determined from

$$s_1 = s_3 + Z_5, \quad s_2 = -s_4 + Z_6 \quad (39)$$

Similarly, the stress function $f_{22}(x,y)$, which corresponds to the symmetric mode, can be expressed in the separable form

$$f_{22}(x,y) = f_a(y) + f_b(y)\cos(2M_2 x) \quad (40)$$

The quantity $f_a(y)_{,yy}$ is found to be

$$f_a(y)_{,yy} = (cM_2^2/2)w_2(y)^2 + c_a \quad (41)$$

where

$$c_a = -cM_2^2 H_2/2 \quad \text{and} \quad H_2 = \int_0^1 w_2(y)^2\,dy \quad (42)$$

The differential equation and the boundary conditions for $f_b(y)$ are identical to that presented for $f_\beta(y)$ by replacing $f_\beta(y)$, $w_1(y)$, and M_1 by $f_b(y)$, $w_2(y)$, and M_2, respectively. Again, exact solution for $f_b(y)$ is sought in the form

$$f_b(y) = P_b(y) + H_b(y) \quad (43)$$

where $H_b(y)$ and $H_\beta(y)$ have the same form and

$$P_b(y) = k_0 + k_1\sin(d_2 y)\sinh(c_2 y) + k_2\cos(d_2 y)\cosh(c_2 y) + k_3\sin(d_2 y)\cosh(c_2 y) + k_4\cos(d_2 y)\sinh(c_2 y) \quad (44)$$

The constant k_0 is found to be

$$k_0 = [(d_2^2 - c_2^2)(A_2)^2 + 2d_2^2(B_2)^2]c/(16M_2^2) \quad (45)$$

and k_1, k_2, k_3, and k_4 can be obtained from

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = (cM_2^2) \begin{bmatrix} 2c_2 d_2 A_2^2 + [c_2 d_2 - (d_2^3/c_2)]B_2^2 \\ -2d_2^2 B_2^2 - (d_2^2 - c_2^2)(A_2)^2 \end{bmatrix}$$

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} = (cM_2^2)(A_2 B_2) \begin{bmatrix} d_2^2 + c_2^2 \\ -c_2 d_2 - (d_2^3/c_2) \end{bmatrix} \quad (46)$$

where

$$R_{11} = R_{22} = c_2^4 + d_2^4 - 6c_2^2 d_2^2 - (8M_2^2)(c_2^2 - d_2^2) + 16M_2^4$$

$$R_{12} = (-1)R_{21} = 4c_2 d_2^3 - 4c_2^3 d_2 + 16c_2 d_2 M_2^2 \quad (47)$$

The constants s_1, s_2, s_3, and s_4 for the homogeneous solution $H_b(y)$ can be obtained from equations (37)–(39) by replacing M_1 and $P_\beta(y)$ by M_2 and $P_b(y)$, respectively.

Finally, the compatibility equation for the second-order stress function f_{12}, which corresponds to the interaction between the antisymmetric and the symmetric mode, is

$$f_{12,xxxx} + f_{12,yyyy} + 2f_{12,xxyy} = (c)[2w_{1,xy}w_{2,xy} - w_{1,xx}w_{2,yy} - w_{1,yy}w_{2,xx}] \quad (48)$$

The solution $f_{12}(x,y)$ can be written in the separable form (letting $M_1 = M_2 = M$ which is valid only for short columns $L/B < 2.3$)

$$f_{12}(x,y) = f_A(y) + f_B(y)\cos(2Mx) \quad (49)$$

In a similar manner, one obtains

$$f_A(y)_{,yy} = (cM^2/2)w_1(y)w_2(y) + c_A$$

where

$$c_A = -cM^2 H_3/2 \quad \text{and} \quad H_3 = \int_0^1 w_1(y)w_2(y)\,dy \quad (50)$$

The differential equation for $f_B(y)$ is

$$f_B(y)_{,yyyy} - (8M^2)f_B(y)_{,yy} + (16M^4)f_B(y) = (M^2)[2w_1(y)_{,y}w_2(y)_{,y} - w_1(y)w_2(y)_{,yy} - w_1(y)_{,yy}w_2(y)] \quad (51)$$

and the boundary conditions for $f_B(y)$ are identical to those applicable for $f_\beta(y)$ presented earlier. Closed-form solution for $f_B(y)$ is obtained in the form

$$f_B(y) = P_B(y) + H_B(y) \quad (52)$$

where $H_B(y)$ has the same form as $H_\beta(y)$ and the exact solution for $P_B(y)$ is

$$P_B(y) = a_1\cosh(c_1 y)\sin(d_2 y) + a_2\sinh(c_1 y)\cos(d_2 y) + a_3\cosh(c_1 y)\cos(d_2 y) + a_4\sinh(c_1 y)\sin(d_2 y) + a_5\cosh(c_1 y)\sinh(c_2 y) + a_6\sinh(c_1 y)\cosh(c_2 y)$$

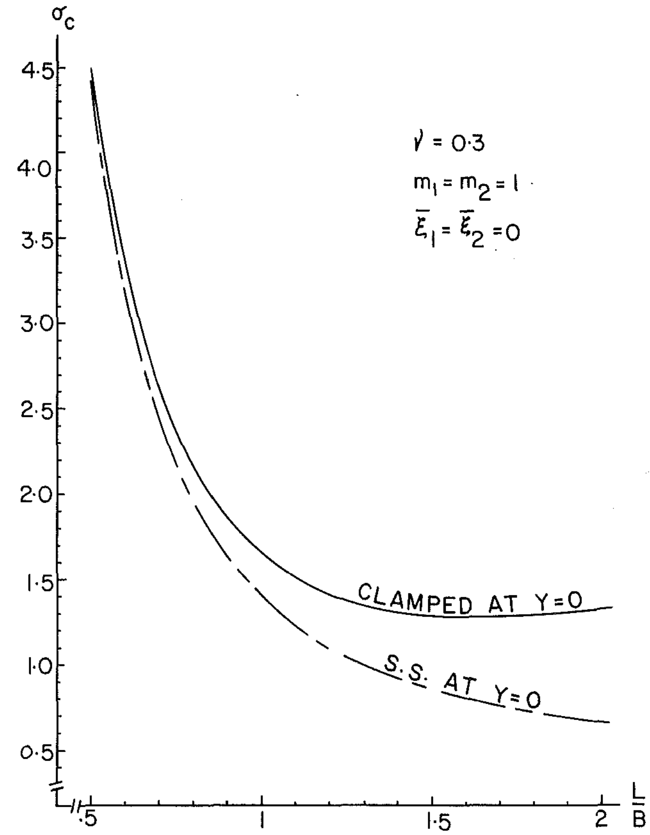

Fig. 2 Antisymmetric and symmetric buckling loads versus the aspect ratio for Poisson's ratio of 0.3

$$+a_7 \cosh(c_1 y) \cosh(c_2 y) + a_8 \sinh(c_1 y) \sinh(c_2 y)$$
$$+a_9 \cos(d_1 y) \sin(d_2 y) + a_{10} \sin(d_1 y) \cos(d_2 y)$$
$$+a_{11} \cos(d_1 y) \cos(d_2 y) + a_{12} \sin(d_1 y) \sin(d_2 y) \quad (53)$$
$$+a_{13} \cos(d_1 y) \sinh(c_2 y) + a_{14} \sin(d_1 y) \cosh(c_2 y)$$
$$+a_{15} \cos(d_1 y) \cosh(c_2 y) + a_{16} \sin(d_1 y) \sinh(c_2 y)$$

For brevity, the lengthy expressions for the coefficients $a_1, a_2, \ldots a_{16}$ and $s_1, \ldots s_4$ will not be shown here. The solutions $f_\beta(y)$, $f_b(y)$, and $f_B(y)$ are verified independently by discretizing the differential equations using a central finite difference scheme with 101 integration points from $y=0$ to $y=1$.

5 Postbuckling Behavior

For a plate, the cubic terms of the potential energy vanish identically so that the potential energy of the present two-mode stability problem is [13]

$$P.E. = \Big\{ (C_{40}\xi_1^4 + C_{04}\xi_2^4 + C_{22}\xi_1^2\xi_2^2) + (C_{31}\xi_1^3\xi_2 + C_{13}\xi_1\xi_2^3) + (D_1/2)[1-(\sigma/\sigma_1)]\xi_1^2 + (D_2/2)[1-(\sigma/\sigma_2)]\xi_2^2 - (\sigma/\sigma_1)\bar{\xi}_1\xi_1 D_1 - (\sigma/\sigma_2)\bar{\xi}_2\xi_2 D_2 \Big\} . \text{constant} \quad (54)$$

In the foregoing, σ_1 and σ_2 are the classical buckling load (of the perfect system) of the antisymmetric and symmetric modes and $\bar{\xi}_1$ and $\bar{\xi}_2$ are the amplitudes of the geometric imperfections in the shapes of these two modes, respectively. The postbuckling coefficients C_{40}, C_{04}, C_{22}, C_{31}, and C_{13} are [18]

$$C_{40} = (1/4)\sigma_{11} \bullet l_2(u_1) = (1/4)\int_0^1 \int_0^{L/B} [f_{11,yy}(w_{1,x})^2 + f_{11,xx}(w_{1,y})^2 - 2f_{11,xy}(w_{1,x}w_{1,y})]dxdy$$
$$C_{04} = (1/4)\sigma_{22} \bullet l_2(u_2)$$
$$C_{22} = (1/4)\sigma_{11} \bullet l_2(u_2) + (1/4)\sigma_{22} \bullet l_2(u_1) + \sigma_{12} \bullet l_{11}(u_1, u_2) \quad (55)$$
$$C_{31} = (1/2)\sigma_{12} \bullet l_2(u_1) + (1/2)\sigma_{11} \bullet l_{11}(u_1, u_2)$$
$$C_{13} = (1/2)\sigma_{12} \bullet l_2(u_2) + \sigma_{22} \bullet l_{11}(u_1, u_2)$$

where, for example,

$$\sigma_{12} \bullet l_{11}(u_1, u_2) = \int_0^1 \int_0^{L/B} [f_{12,yy}(w_{1,x}w_{2,x}) + f_{12,xx}(w_{1,y}w_{2,y}) - f_{12,xy}(w_{1,x}w_{2,y} + w_{1,y}w_{2,x})]\, dxdy \quad (56)$$

Further, the positive quantities D_1 and D_2 are defined to be

$$D_1 = (-1)f_{0,yy}(\text{at} \quad \sigma=\sigma_1)\int_0^1 \int_0^{L/B} (w_{1,x})^2 dx\, dy$$
$$D_2 = (-1)f_{0,yy}(\text{at} \quad \sigma=\sigma_2)\int_0^1 \int_0^{L/B} (w_{2,x})^2 dx\, dy \quad (57)$$

Minimizing the potential energy with respect to the amplitudes of the buckling modes ξ_1 and ξ_2, the two equilibrium equations are

$$b_1\xi_1^3 + b_{12}\xi_1\xi_2^2 + C_{311}\xi_1^2\xi_2 + C_{131}\xi_2^3 + [1-(\sigma/\sigma_1)]\xi_1 = (\sigma/\sigma_1)\bar{\xi}_1$$
$$b_2\xi_2^3 + b_{21}\xi_1^2\xi_2 + C_{312}\xi_1^3 + C_{132}\xi_1\xi_2^2 + [1-(\sigma/\sigma_2)]\xi_2 = (\sigma/\sigma_2)\bar{\xi}_2 \quad (58)$$

where

$$b_1 = 4C_{40}/D_1 \quad b_2 = 4C_{04}/D_2, \quad b_{12} = 2C_{22}/D_1, \quad b_{21} = 2C_{22}/D_2$$
$$C_{311} = 3C_{31}/D_1 \quad C_{131} = C_{13}/D_1, \quad C_{312} = C_{31}/D_2, \quad C_{132} = 3C_{13}/D_2 \quad (59)$$

In computing these postbuckling coefficients, integration in the axial x direction can be carried out analytically while integration in the y direction is performed by computing the area using Simpson's rule.

As a check on the preceding coefficients, it can be seen that the foregoing two equilibrium equations agree with those presented by Byskov and Hutchinson [8, 18–20],

$$b_{1111}\xi_1^3 + (b_{1221} + b_{2121} + b_{2211})\xi_1\xi_2^2 + (b_{1121} + b_{1211} + b_{2111})\xi_1^2\xi_2 + b_{2221}\xi_2^3 + [1-(\sigma/\sigma_1)]\xi_1 = (\sigma/\sigma_1)\bar{\xi}_1$$
$$b_{2222}\xi_2^3 + (b_{1122} + b_{1212} + b_{2112})\xi_1^2\xi_2 + b_{1112}\xi_1^3 + (b_{1222} + b_{2122} + b_{2212})\xi_1\xi_2^2 + [1-(\sigma/\sigma_2)]\xi_2 = (\sigma/\sigma_2)\bar{\xi}_2 \quad (60)$$

where

$$b_{ijkJ} = [\sigma_J \bullet l_{11}(u_j, u_k) + \sigma_{ij} \bullet l_{11}(u_k, u_J)]/(2D_J) \quad (61)$$

It should be noted in the isotropic plate buckling problem, $\sigma_J \bullet l_{11}(u_j, u_k)$ is zero.

In the present two-mode stability problem, it is found that $C_{13} = C_{31} = 0$. For a column with a purely symmetric imperfection ($\bar{\xi}_1 = 0$ and $\bar{\xi}_2 \neq 0$), the equilibrium path will follow the mode shape of a symmetric mode until the applied load is sufficiently large such that it will intersect the stability boundary and thus, bifurcation into the antisymmetric mode will occur. This equilibrium path is governed by

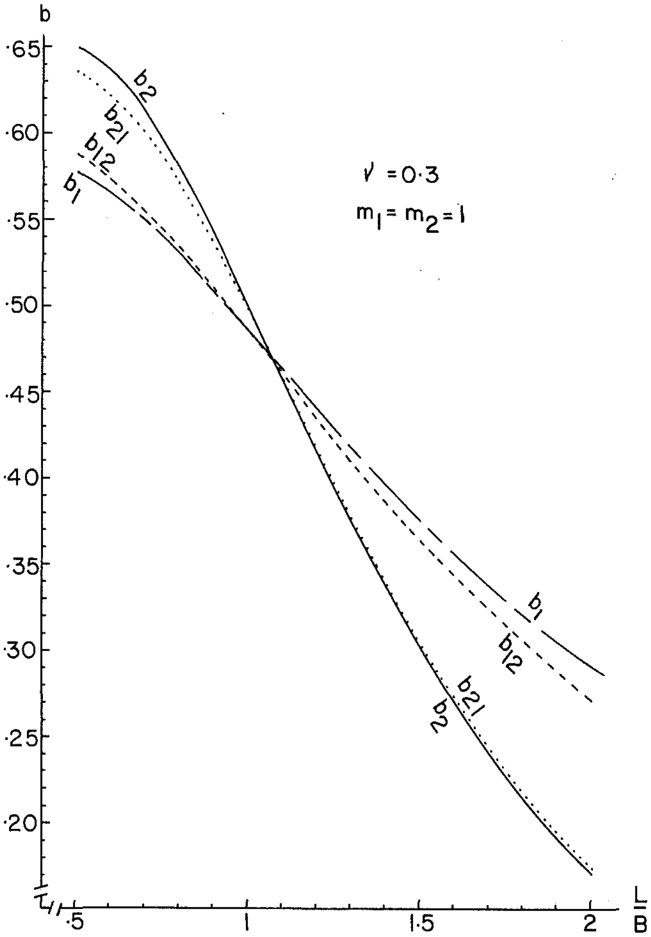

Fig. 3 The b_1, b_2, and b_{22} coefficients versus the aspect ratio for Poisson's ratio of 0.3

$$b_2\xi_2^3 + [1-(\sigma/\sigma_2)]\xi_2 = (\sigma/\sigma_2)\bar{\xi}_2 \tag{62}$$

and the stability boundary is specified by

$$b_{12}\xi_2^2 + [1-(\sigma/\sigma_1)] = 0 \tag{63}$$

Note that this particular problem is independent of the b_1 and b_{21} coefficients.

6 Discussion of Results

Figure 2 shows a graph of the buckling load versus the aspect ratio of the plate L/B where Poisson's ratio is $\nu = 0.3$. Within the range of the values of L/B being 0.5 and 2, the classical buckling loads σ_1 and σ_2 correspond to one half sine-wave in the longitudinal direction ($m_1 = m_2 = 1$). It can be seen that the symmetric mode (clamped at $y = 0$) has a higher buckling load than the antisymmetric mode (simply supported at $y = 0$). The difference between these two buckling loads becomes smaller for shorter columns.

Figure 3 shows a plot of the postbuckling coefficients b_1, b_2, b_{12} and b_{21} versus the aspect ratio of the plate. It can be seen that all these coefficients are found to be positive so that the postbuckling problem is stable. That is, buckling will not be associated with the loss of the carrying capacity of the structure, at least in the initial postbuckling regime. These coefficients become more positive for shorter columns. It should be noted that C_{13} and C_{31} vanish for the present range of L/B being considered.

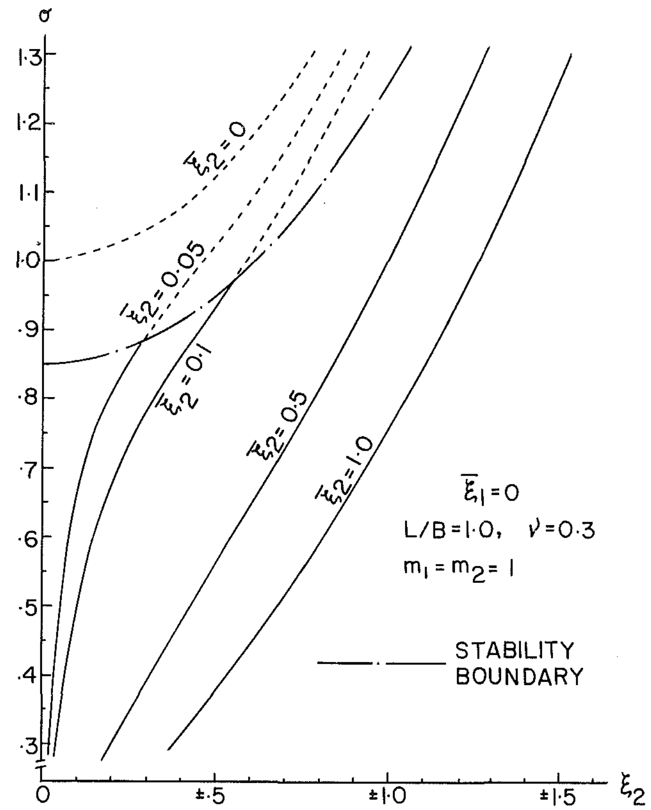

Fig. 4 Equilibrium paths of a symmetrically imperfect column with $L/B = 1.0$ and $\nu = 0.3$

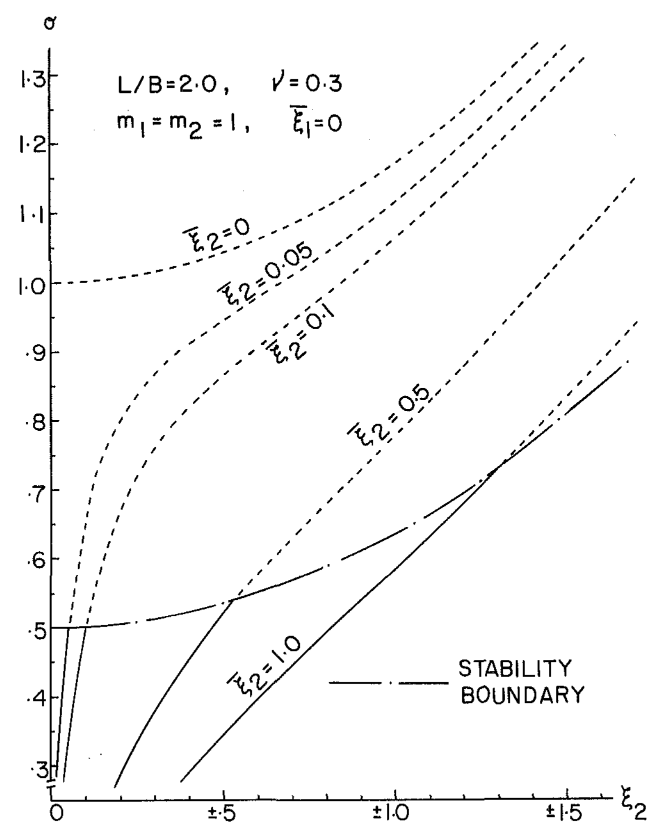

Fig. 5 Equilibrium paths of a symmetrically imperfect column with $L/B = 2.0$ and $\nu = 0.3$

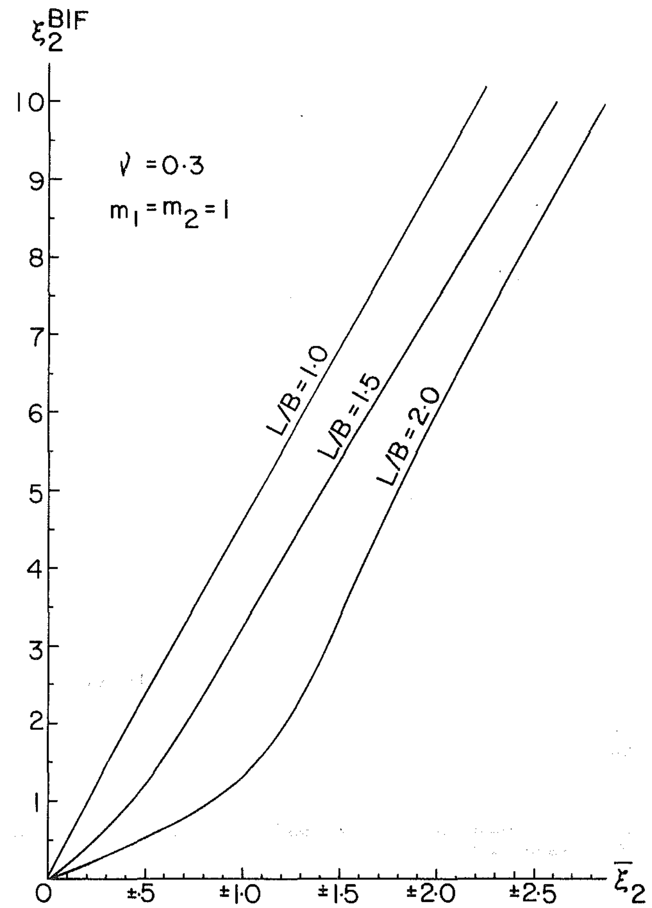

Fig. 6 Amplitude of the symmetric mode (at which bifurcation into the antisymmetric mode just occurs) versus amplitude of the symmetric imperfection

The aim of the present study is to examine the possibility of introducing a geometric imperfection in the shape of the symmetric mode so that the structure will follow the equilibrium path of this mode in the initial postbuckling regime as well as in the very large deflection stage. Thus, it is of interest to examine the equilibrium path of the column with $\bar{\xi}_1 = 0$ and $\bar{\xi}_2 \neq 0$. Figure 4 shows a graph of the applied load versus the amplitude of the symmetric mode ξ_2 for $L/B = 1$ and $\nu = 0.3$, where $\sigma_1/\sigma_2 = (1.4016/1.6525) = 0.8481$, $b_2 = 0.500$, and $b_{12} = 0.4835$. The structure is assumed to be loaded from zero so that the solid lines are of physical significance while the dotted lines are unstable equilibrium paths (physically unattainable in general). As the applied load is increased the stable equilibrium paths of an imperfect system will eventually intersect the stability boundary, at which bifurcation into the antisymmetric mode will occur.

Figure 5 shows the equilibrium path of an imperfect column ($\bar{\xi}_1 = 0$ and $\bar{\xi}_2 \neq 0$) for $L/B = 2$ and $\nu = 0.3$ where $\sigma_1/\sigma_2 = (0.6681/1.3360) = 0.500$, $b_2 = 0.1716$ and $b_{12} = 0.2719$. Since both the b_2 and b_{12} coefficients for $L/B = 2$ are smaller than those for $L/B = 1$, the perfect path $\bar{\xi}_2 = 0$ and the stability boundary opens wider. Thus, a larger value of the imperfection amplitude $\bar{\xi}_2$ is needed to avoid bifurcation into the antisymmetric mode. Note that the present Koiter-type analysis is valid only for sufficiently small values of the imperfection amplitude. However, the initial postbuckling behavior is crucial in determining the final collapse mode.

Figure 6 depicts a graph of the value of the amplitude of the symmetric mode (at which bifurcation into the antisymmetric mode just occurs) versus the symmetric imperfection amplitude $\bar{\xi}_2$ for aspect ratio L/B being 1.0, 1.5, and 2.0. As predicted from Fig. 4, a larger value of $\bar{\xi}_2$ is needed to achieve a symmetric mode shape with $\xi_2 = 10.0$ without bifurcation into the antisymmetric mode. For $L/B = 1.5$, it is found that $\sigma_1/\sigma_2 = (0.8578/1.2912) = 0.6643$, $b_2 = 0.3031$ and $b_{12} = 0.3633$.

7 Concluding Remarks

The two-mode stability problem which involved the antisymmetric and the symmetric modes for a short, thin-walled, geometrically imperfect column has been investigated. The postbuckling coefficients of this two nonsimultaneous mode problem are computed. It appears from the equilibrium paths that it is possible to introduce a geometric imperfection in the shape of a symmetric mode so that the equilibrium paths will follow the shape of a symmetric mode in the postbuckling as well as in the large deflection regimes. This design is particularly beneficial from the energy absorption point of view since the symmetric mode has a much higher energy absorption than the antisymmetric mode.

Finally, the plastic collapse analysis of short, imperfect thin-walled angled columns (taking into account plastic deformation in the postbuckling range) will be examined in a separate paper.

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