$$p = -\frac{\alpha C}{K} \pm \frac{i}{K}(KC\beta - \alpha^2 C^2)^{1/2}. \tag{35}$$

Note that in (35) the ellipticity condition $\lambda_{11}\lambda_{22} > \lambda_{12}^2$ ensures that $KC\beta - \alpha^2 C^2 > 0$.

Hence from (33), (18), and (27) it follows that if λ_{12}, λ_{11} and λ_{22} are given by (28)–(30) then the solution to (4) may be written in the form

$$T = (Cy + D)^{-1}\Re[\psi(z)] \tag{36}$$

where ψ is an arbitrary analytic function, $z = x + py$, with x, y, and p defined, respectively, by (22), (21), and (35).

A Specific Boundary Value Problem

Complex function theory may be used in conjunction with the solution (36) to provide the solution to a number of thermostatic boundary value problems for inhomogeneous materials. As an illustration a particular boundary value problem will be examined in this section.

Consider the anisotropic half space occupying the region $x_2 > 0$ and let the temperature distribution on the boundary $x_2 = 0$ of the half space be given by

$$T(x_1, 0) = \begin{cases} T_0(x_1) & \text{for} \quad |x_1| < a \\ 0 & \text{for} \quad |x_1| > a \end{cases} \tag{37}$$

where $T_0(x_1)$ is a given function of x_1. Suppose further that the half space is inhomogeneous and is of a form that, for suitable values of the constants, the coefficients λ_{11}, λ_{12}, and λ_{22} are given by (28)–(30). It is required to find the temperature throughout the half space.

The temperature in the half space may be represented by the form (36) with the analytic function $\psi(z)$ satisfying the boundary condition

$$D^{-1}\Re[\psi(x)] = \begin{cases} T_0(x) & \text{for} \quad |x_1| < a \\ 0 & \text{for} \quad |x_1| > a \end{cases} \tag{38}$$

Thus it is required to find function $\psi(z)$ which is analytic in the upper half space $x_2 > 0$, vanishes at infinity and satisfies the boundary condition (38). This is a standard problem (see for example Clements [2]) and the solution is

$$\psi(z) = \frac{1}{2\pi i}\int_{-a}^{a} \frac{DT_0(t)}{t - z}. \tag{39}$$

Hence, from (36), the temperature in the half space is given by

$$T = (Cy + D)^{-1}\Re\left\{\frac{1}{2\pi i}\int_{-a}^{a}\frac{DT_0(t)}{t - z}\right\}. \tag{40}$$

In the particular case when $T_0(t) = t_0$ (constant) equation (40) provides

$$T = (Cy + D)^{-1}Dt_0\Re\left\{\frac{1}{2\pi i}\ln\left[\frac{z - a}{z + a}\right]\right\}. \tag{41}$$

Consider now some particular cases for the function $k(x_2)$ occurring in (21)

(*i*) $k(x_2) = 1$. In this case $y = x_2$ and hence (40) and (28)–(30) yield

$$T = (Cx_2 + D)^{-1}Dt_0\Re\left\{\frac{1}{2\pi i}\ln\left[\frac{z - a}{z + a}\right]\right\} \tag{42}$$

with $z = x + py = x_1 + px_2$ and

$$\lambda_{12} = \alpha(Cx_2 + D)^2, \quad \lambda_{22} = KC^{-1}(Cx_2 + D)^2,$$
$$\lambda_{11} = \beta(Cx_2 + D)^2. \tag{43}$$

Thus if the conductivities λ_{ij} increase with x_2 as prescribed by (43) then the temperature as given by (42) decreases with x_2 more rapidly than would be the case with constant conductivities ($C = 0$).

(*ii*) $k(x_2) = ae^{bx_2}$ $(b > 0)$. Here $y = a/b(e^{bx_2} - 1)$ so that

$$T = (Cab^{-1}e^{bx_2} - Cab^{-1} + D)^{-1}Dt_0\Re\left\{\frac{1}{2\pi i}\ln\left[\frac{z - a}{z + a}\right]\right\}$$

with $z = x_1 + pab^{-1}(e^{bx_2} - 1)$ and

$$\lambda_{12} = \alpha[Cab^{-1}(e^{bx_2} - 1) + D]^2, \quad \lambda_{22} = K[Cab^{-1}(e^{bx_2} - 1) + D]^2/Cae^{bx_2},$$
$$\lambda_{11} = \beta ae^{bx_2}[Cab^{-1}(e^{bx_2} - 1) + D]^2.$$

(*iii*) $k(x_2) = a(1 + \cos bx_2)$. In this case $y = a(x_2 + b^{-1}\sin bx_2)$

$$T = [Ca(x_2 + b^{-1}\sin bx_2) + D]^{-1}Dt_0\Re\left\{\frac{1}{2\pi i}\ln\left[\frac{z - a}{z + a}\right]\right\}$$

with $z = x_1 + pa(x_2 + b^{-1}\sin bx_2)$ and

$$\lambda_{12} = \alpha[Ca(x_2 + b^{-1}\sin bx_2) + D]^2,$$
$$\lambda_{22} = K[Ca(x_2 + b^{-1}\sin bx_2) + D]^2/Ca(1 + \cos bx_2),$$
$$\lambda_{11} = \beta a(1 + \cos bx_2)[Ca(x_2 + b^{-1}\sin bx_2) + D]^2.$$

These three examples give some indication of the possible types of variations in the λ_{ij} for which an elementary general solution of (2.1) exists. In general the aim would be to choose a suitable multiparameter form for $k(x_2)$ so that the λ_{ij} could be fitted to given numerical values.

References

1 Carslaw, H. S., and Jaeger, J. C., *Conduction of Heat in Solids*, Clarendon Press, Oxford, 1959.

2 Clements, D. L., *Boundary Value Problems Governed by Second Order Elliptic Systems*, Pitman, London, 1981.

Effects of Geometric Imperfections on Large-Amplitude Vibrations of Rectangular Plates With Hysteresis Damping

David Hui[1]

The present paper deals with the effects of initial geometric imperfections on large-amplitude vibrations of simply supported rectangular plates. The vibration mode, the geometric imperfection, and the forcing function are taken to be of the same spatial shape. It is found that geometric imperfections of the order of a fraction of the plate thickness may significantly raise the free linear vibration frequencies. Furthermore, contrary to the commonly accepted theory that large-amplitude vibrations of plates are of the hardening type, the presence of small geometric imperfections may cause the plate to exhibit a soft-spring behavior. The effects of hysteresis (structural) damping on the vibration amplitude are also examined.

[1] Assistant Professor, Department of Engineering Mechanics, The Ohio State University, Columbus, Ohio 43210. Mem. ASME.

Manuscript received by ASME Applied Mechanics Division, September, 1982; final revision, July, 1983.

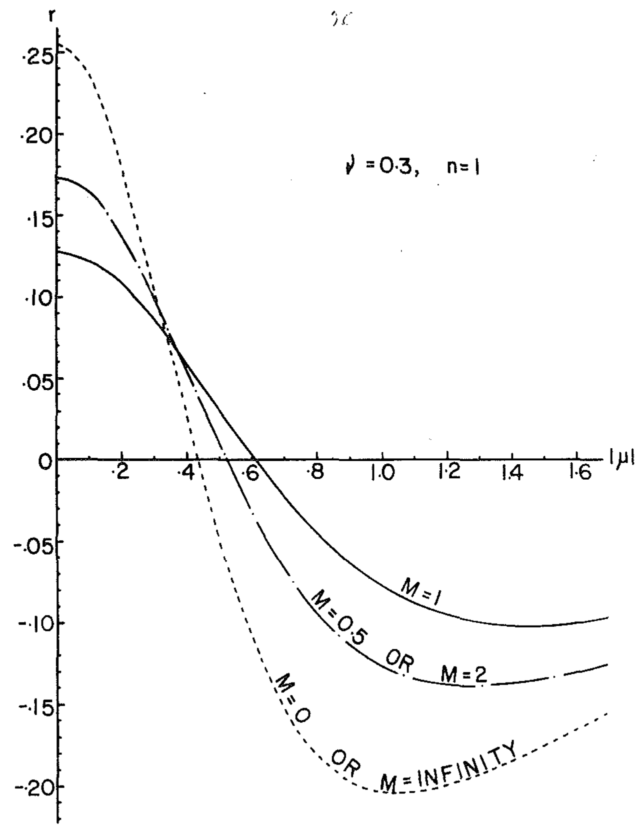

Fig. 1(*a*) Nonlinearity parameter versus imperfection amplitude for simply supported rectangular plates with $\nu = 0.3$ and $n = 1$

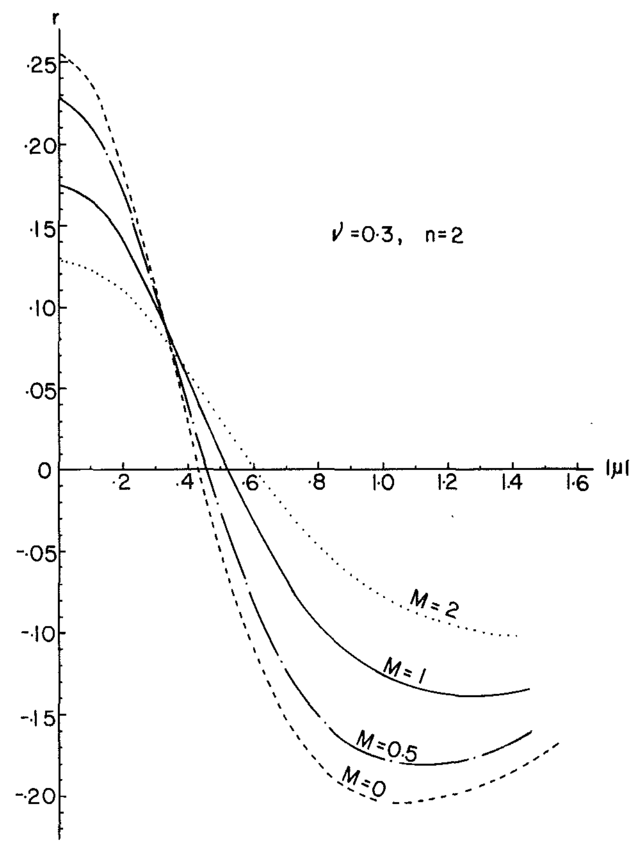

Fig. 1(*b*) Nonlinearity parameter versus imperfection amplitude for simply supported rectangular plates with $\nu = 0.3$ and $n = 2$

Fig. 1

1 Introduction

Large-amplitude vibration of isotropic homogeneous plates is an important subject which has been investigated by many authors in the past decades [1-4] and ân excellent introduction to this topic can be found in a book by Chia [5]. Many other relevant papers are cited by the foregoing references. It was found that the large amplitude vibrations of plates were always of the hardening type for various types of boundary conditions and shapes of the plates.

Much less is known about the effects of initial geometric imperfections on linear or nonlinear vibrations of plates. Hui and Leissa [6] examined the effects of geometric imperfections on linear free vibrations of simply supported rectangular plates with two in-plane biaxial preloaded forces. They found that initial geometric imperfections of the order of the plate thickness may significantly raise the vibration frequencies, especially in the region of large in-plane preload. Large-amplitude vibrations of clamped circular plates with initial geometric imperfections and initial in-plane edge displacements was investigated by Yamaki et al. [4]; however, the results are restricted to only one value of the geometric imperfection amplitude being 10 percent of the plate thickness and only hard-spring behavior was found. A reexamination of the large-amplitude vibrations of circular plates shows that the geometric imperfections of the order of half the plate thickness may significantly raise the free linear vibration frequencies and cause the circular plates to exhibit soft-spring behavior [7]. The large-amplitude vibration behavior of imperfect plates was examined earlier [8]; but the results are confined to square plates with only one half-wave in both directions and the imperfection amplitudes are too large to be considered practically unavoidable. In passing, the effects of geometric imperfections on linear vibration of shallow spherical shells with pressurized preload [9] and nonlinear vibrations of shallow spherical shells [10] have been examined. The important related work on the cylindrical panels [11] and cylindrical shells is in progress.

The present paper aims to examine the effects of geometric imperfections on large-amplitude vibrations of simply supported rectangular plates. The vibration mode, the initial geometric imperfection, and the forcing function are all of the same spatial shape. The effects of structural damping are studied using the complex-modulus model [12]. The analysis is based on a solution of the nonlinear von Kármán differential equations for plates. The nonlinear compatibility equation is satisfied exactly by a suitable choice of the stress function, and the nonlinear dynamic equilibrium equation is satisfied approximately using a Galerkin procedure. The resulting second-order nonlinear ordinary differential equation in time can be written in the form of Duffing's equation with an additional quadratic-spring term.

It is found that the presence of geometric imperfection may significantly raise the free vibration frequency. Contrary to the well-established and widely accepted theory that the nonlinear vibration of flat plates is of the hardening type, the present analysis shows that the presence of unavoidable geometric imperfection amplitudes of the order of only half the plate thickness may change the nonlinear hard-spring character of the plate to one with a soft-spring behavior. Appropriate backbone curves are plotted. Structural damping is shown to reduce significantly the forced linear vibration amplitude of the plate near resonance.

2 Analysis

The dynamic analogue of von Karman equilibrium and compatibility differential equations for moderately large-amplitude vibrations of plates, written in terms of the out-of-plane displacement w and the Airy stress function f, in-

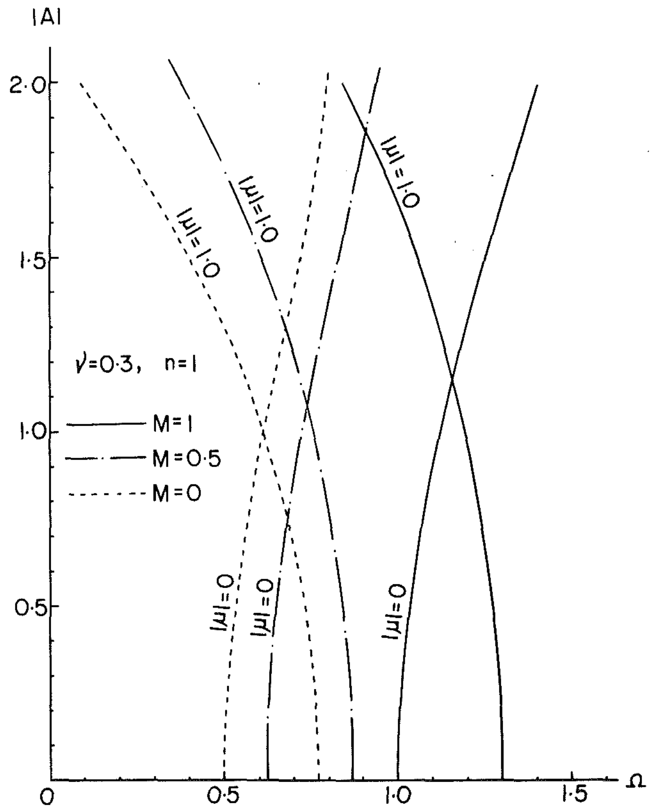

Fig. 2(*a*) Backbone curves for free nonlinear vibration of simply supported imperfect rectangular plates with no damping ($|\mu|=0$, 1.0, $\nu=0.3$, and $n=1$)

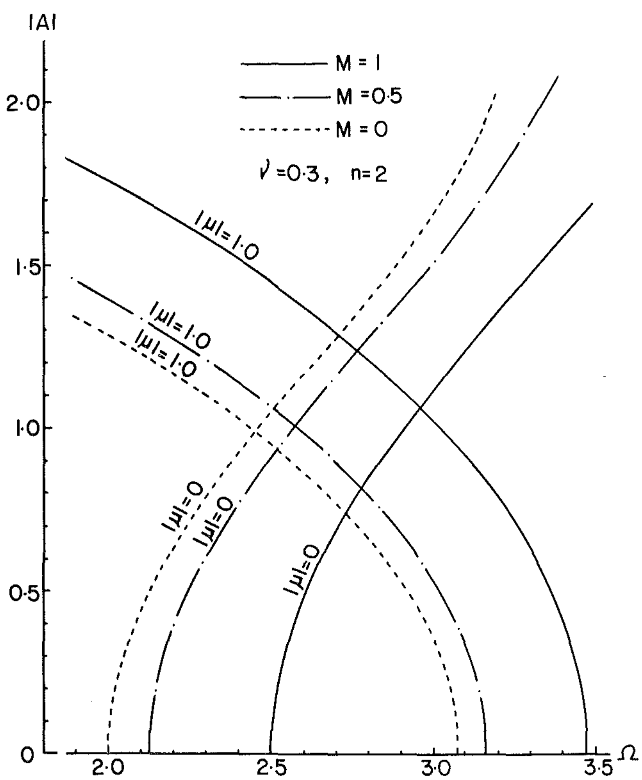

Fig. 2(*b*) Backbone curves for free nonlinear vibration of simply supported imperfect rectangular plates with no damping ($|\mu|=0$, 1.0, $\nu=0.3$, and $n=2$)

Fig. 2

corporating the possibility of a geometric imperfection w_0 are, respectively, in nondimensional form [5, 11, 13],

$$(1+i\eta)(w,_{xxxx}+w,_{yyyy}+2w,_{xxyy}) = 4\pi^4 q(x,y,t) - 4\pi^4 w,_{tt} + (2c)[f,_{xx}(w+w_0),_{yy} + f,_{yy}(w+w_0),_{xx} - 2f,_{xy}(w+w_0),_{xy}] \quad (1)$$

$$[1/(1+i\eta)](f,_{xxxx}+f,_{yyyy}+2f,_{xxyy}) = (2c)[(w,_{xy})^2 + 2w_{0,xy}w,_{xy} - (w+w_0),_{xx}w,_{yy} - w_{0,yy}w,_{xx}] \quad (2)$$

where,

$$(w,w_0)=(W,W_0)/h, \quad f=2cF/(\mathrm{E}h^3),$$
$$(x,y)=(X,Y)/b,$$
$$t=\bar{t}\omega_r, \quad q(x,y,t)=[c^2b^4/(\mathrm{E}h^4\pi^4)]Q(X,Y,\bar{t})$$
$$(\omega_r)^2 = 4\pi^4 D/(\rho b^4) = \pi^4 \mathrm{E}h^3/(\rho c^2 b^4) \quad (3)$$

In the foregoing, E is Young's modulus, η is the loss factor associated with the complex-modulus model for structural damping, $i=(-1)^{1/2}$, h is the plate thickness, W_o is the initial geometric imperfection, X and Y are the in-plane coordinates, $Q(X,Y,\bar{t})$ is the forcing function, ρ is the plate mass per unit area, $\bar{t}$ is the time, $c=[3(1-\nu^2)]^{1/2}$, ν is Poisson's ratio, b is the width of the plate, and ω_r is the reference frequency.

The boundary conditions are taken to be simply supported of the form (a is the length of the plate)

$$w(x=0 \text{ or } a/b)=0$$
$$w,_{xx}(x=0 \text{ or } a/b)=0$$
$$w(y=0 \text{ or } 1)=0$$
$$w,_{yy}(y=0 \text{ or } 1)=0 \quad (4)$$

Further, the in-plane displacements normal to the edges are constant and there is no in-plane shear along all the edges, that is, $f,_{xy}(x=0 \text{ or } x=a/b)=0$ and $f,_{xy}(y=0 \text{ or } y=1)=0$.

For a rectangular plate, the fundamental vibration mode corresponds to half sine-waves in both the x and y directions. From a physical point of view, it is realistic to assume that the shape of the geometric imperfection (unavoidable deviation from flatness) is the same as that of the fundamental mode. This will serve as an example problem to demonstrate that the large-amplitude vibration of imperfect plates is not necessarily of the hardening type. The influence of geometric imperfections (of shapes other than the vibration mode) on small-amplitude vibrations of plates was examined [6]. However, their influence on large-amplitude vibrations of plates will not be presented here. The shape of the forcing pressure distribution is taken to be of the same shape as the vibration mode; the specification of this type of pressure distribution will not affect the problem since the present paper deals primarily with free vibrations of plates. Accordingly, the vibration mode, the initial geometric imperfection, and the forcing pressure distributions are taken to be the same spatial shape,

$$[w(x,y,t), w_0(x,y), q(x,y,t)] = [w(t), \mu, q_1\cos(\omega t/\omega_r)]\sin(M\pi x)\sin(n\pi y) \quad (5)$$

where μ is the amplitude of the geometric imperfection normalized with respect to the plate thickness, $M=mb/a$, and m and n are the number of half-waves in the x and y directions, respectively. Substituting $w(x,y,t)$ and $w_0(x,y)$ into equation (2), the stress function that satisfies the boundary conditions and the nonlinear compatibility equation exactly is,

$$f(x,y,t) = (1+i\eta)[w(t)^2 + 2\mu w(t)](c/16) [(n/M)^2\cos(2M\pi x) + (M/n)^2\cos(2n\pi y)] \quad (6)$$

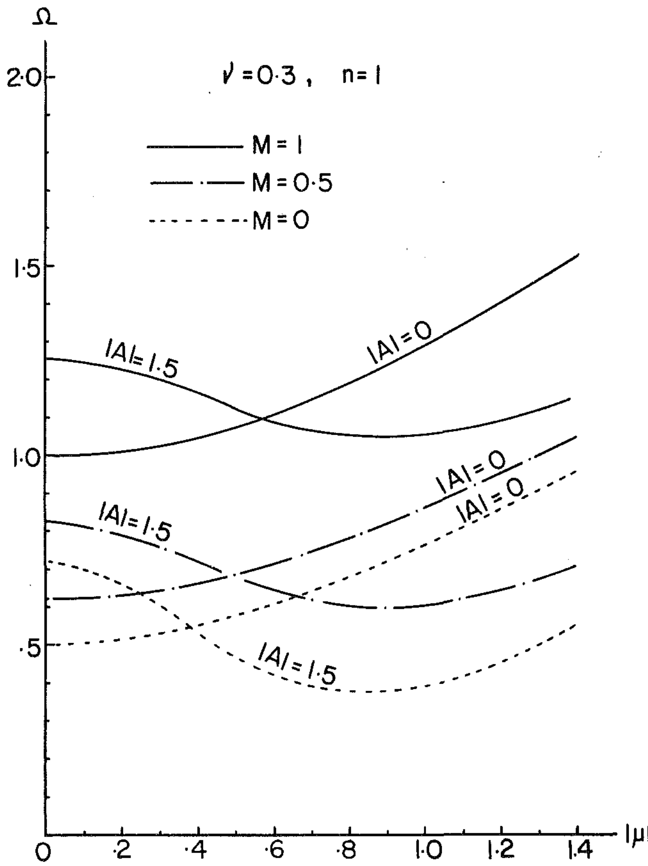

Fig. 3 Free vibration frequency versus imperfection amplitude for simply supported rectangular plates with $|A| = 0, 1.5$, $\nu = 0.3$, and $n = 1$

Finally, substituting $w(x,y,t)$, $w_0(x,y)$ and $f(x,y,t)$ into the nonlinear dynamic equilibrium equation and applying the Galerkin procedure (i.e., multiplication of both sides by $\sin(M\pi x)\ \sin(n\pi y)$ followed by integration over the plate surface) yields a second-order nonlinear ordinary differential equation in time which can be written in the form of Duffing's equation with the additional quadratic term,

$$w(t)_{,tt} + [kw(t) + (\epsilon k a_2) w(t)^2 + (\epsilon k) w(t)^3](1 + i\eta) = q_1 \cos(\omega t/\omega_r) \quad (7)$$

where k, ϵ, and a_2 are found to be,

$$k = [(M^2+n^2)^2 + (M^4+n^4)(\mu^2 c^2/2)](1/4)$$
$$\epsilon = (M^4+n^4)[c^2/(16k)] \quad (8)$$
$$a_2 = 3\mu$$

Thus, the solution of the linearized differential equation is,

$$w(t) = A\cos(\omega t/\omega_r) \quad (9)$$

where the absolute value of the complex quantity A is,

$$|A| = \frac{(q_1/k)}{\{[1-(\Omega^2/k)]^2 + \eta^2\}^{1/2}} \quad (10)$$

Further, using Linstedt's perturbation technique (see the Appendix of [7]), the ratio of the nonlinear to the linear vibration frequency Ω/Ω_0 is related to the vibration amplitude A in the form (assuming no damping, $\eta = 0$),

$$\Omega/\Omega_0 = 1 + rA^2 - (15\epsilon^2 A^4/256) \quad (11)$$

where

$$r = (3\epsilon/8) - (5{a_2}^2\epsilon^2/12) \quad (12)$$

Thus, the sign of the nonlinearity parameter r determines whether the plate behaves (at least for small values of the amplitude A) as a hard-spring or a soft-spring. Clearly, a sign change in the imperfection amplitude will not affect the problem. Setting r to zero, the transition imperfection amplitude is,

$$\mu_t = \frac{M^2+n^2}{[(2c)^2(M^4+n^4)]^{1/2}} \quad (13)$$

so that for the amplitude of the geometric imperfection greater than μ_t, the nonlinear vibrations of simply supported rectangular plates is of the softening type.

3 Results and Discussion

The influence of structural damping on the linear forced vibration response (for practical values of the loss factor [12] η being 0.1, 0.01, and 0.001) is summarized in Table 1. The amplitude is normalized with respect to its static value ($|Ak/q_1|$) while the forcing frequency is normalized ($\Omega = \omega/\omega_r$) with respect to the linear vibration frequency ($\Omega/\sqrt{k}$). It can be seen that the peaks of the response curves remains unshifted at $\Omega/\sqrt{k} = 1$. Also, the peak value is equal to $1/\eta$.

Figure 1(a) shows a graph of the nondimensional nonlinearity parameter r versus the amplitude of the geometric imperfection for simply supported rectangular plates with Poisson's ratio $\nu = 0.3$ and wave numbers $n = 1$ and $M = mb/a = 0$, 0.5, 1, 2, and infinity. Since the influence of the quartic term A^4 in equation (11) is generally small (it is negligible compared with the A^2 term for small A), the large-amplitude vibration problem can be classified as a hard-spring or a soft-spring type depending on whether r is positive or negative, respectively. These curves show that the nonlinearity parameter r decreases and then increases slightly with increasing imperfection amplitude. A similar plot for the wave number $n = 2$ is presented in Fig. 1(b). The transitional values of the imperfection amplitude at which the nonlinearity parameter r changes sign can be computed using equation (13); for $n = 1$ or $n = 2$, $|\mu_t|$ lies between 0.42 and 0.60 for all possible values of M. Finally, the aspect ratio of the plate affects the problem via the wave number M.

The backbone curves (magnitude of the vibration amplitude $|A|$ versus free nonlinear vibration frequency Ω) for simply supported rectangular plates with no damping are shown in Fig. 2(a). Again, Poisson's ratio ν is taken to be 0.3 and the wave numbers are $n = 1$ and $M = mb/a = 0$, 0.5, and 1.0. It can be seen that for perfect plates ($\mu = 0$), the nonlinear vibration is of the hardening type. Note that although the geometric imperfection raises the free linear vibration frequency (at $|A| = 0$), the nonlinear frequency may be signficantly reduced as a result of the nonlinear softening character of the imperfect plate. A similar plot for the wave number $n = 2$ is depicted in Fig. 2(b). Note that in order to show the preceding effects, the nonlinearity frequency Ω is not normalized with respect to its linear value.

Figure 3 shows the nonlinear frequency versus the imperfection amplitude for simply supported rectangular plates with $\nu = 0.3$, $n = 1$, and $M = 0$, 0.5, and 1.0 for fixed values of the magnitude of the vibration amplitude $|A| = 0$ and 1.5. Again, the nondimensional frequency Ω, defined to be ω/ω_r, is not to be confused with ω(nonlinear)/ω(linear). Whereas the effect of geometric imperfection is to raise free vibration frequency as seen in the $|A| = 0$ curves, the effect of imperfection may actually lower the nonlinear frequency for a fixed value of the magnitude of the vibration amplitude, say $|A| = 1.5$. It should be noted that the eventual rise of the nonlinear frequency for $|A| = 1.5$ is attributed to two factors: the nonlinearity parameter r becomes slightly less negative as seen in Fig. 1(a) and the linear vibration frequency continues to rise with increasing magnitude of the geometric im-

Table 1 Normalized amplitude ($|Ak/q_1|$) versus normalized frequency ($\Omega/\sqrt{k}$) for a given loss factor

$\Omega/(k^{1/2})$	$\lvert Ak/q_1\rvert$ (at $\eta=0.1$)	$\lvert Ak/q_1\rvert$ (at $\eta=0.01$)	$\lvert Ak/q_1\rvert$ (at $\eta=0.001$)
0.00	0.995037	0.999950	0.999999
0.20	1.036060	1.041610	1.041660
0.40	1.182130	1.190390	1.190470
0.60	1.543770	1.562310	1.562480
0.80	2.676440	2.776710	2.777670
0.85	3.390200	3.601270	3.603370
0.90	4.657460	5.255880	5.262430
0.95	7.160000	10.202900	10.251000
0.975	8.966580	19.850100	20.211800
1.00	10.000000	100.000000	1000.000000
1.025	8.921860	19.378600	19.714700
1.05	6.983240	9.710000	9.751460
1.10	4.299340	4.756510	4.761370
1.15	2.961660	3.099290	3.100630
1.20	2.216210	2.272140	2.272670
1.40	1.036060	1.041610	1.041660
1.60	0.639713	0.641013	0.641024
1.80	0.445984	0.446424	0.446428
2.00	0.333148	0.333332	0.333333
2.50	0.190442	0.190476	0.190476
3.00	0.124990	0.125000	0.125000
4.00	0.0666652	0.0666667	0.0666667
5.00	0.0416663	0.0416667	0.0416667
7.50	0.0180995	0.0180995	0.0180995
10.00	0.0101010	0.0101010	0.0101010

perfection. For brevity, a similar plot for the $n=2$ case will not be presented here.

4 Concluding Remarks

The effects of geometric imperfections on the linear and nonlinear vibrations of simply supported rectangular plates were examined. It was found that the presence of geometric imperfections of the order of a fraction of the plate thickness may significantly raise the free linear vibration frequencies. Furthermore, small geometric imperfections may drastically change the nonlinear character of the vibration problem from the well-accepted hard-spring behavior to one with a soft-spring behavior. The soft-spring behavior of nonlinear axisymmetric vibrations of clamped shallow spherical caps with circular edges [14] supports qualitatively the soft-spring findings of the present paper for the special case when $M=m=n=a/b=1$.

Finally, it should be mentioned that the geometric imperfection is taken to be of the same shape as the vibration mode. Analogous to the buckling and initial postbuckling problems [15], it is quite possible that this type of geometric imperfection is the most influential one. That is, if the effects of this type of imperfection are not significant, then it is probably not worthwhile to consider other types of imperfection. The severe effects of geometric imperfections in the present analysis show that the influence of unavoidable random imperfections cannot be "safely" neglected in practical designs. While it is highly desirable to obtain more accurate hard-spring curves (using the amplitude incremental method [16] and including the effects of shear and rotatory inertia [17]), it is equally important to investigate whether practical imperfect plates behave as hard-spring or soft-spring in large-amplitude vibrations. Further work on the effects of geometric imperfections on large-amplitude vibrations of angle and cross-ply composite plates will be presented in separate papers.

5 Acknowledgment

The work was started while the author was a graduate student at the Department of Ocean Engineering, Massachusetts Institute of Technology, Cambridge, Mass. 02139.

References

1 Chu, H. N., and Herrmann, G., "Influence of Large Amplitudes on Free Flexural Vibrations of Rectangular Elastic Plates," ASME JOURNAL OF APPLIED MECHANICS, Vol. 23, 1956, pp. 532–540.

2 Yamaki, N., "Influence of Large Amplitudes on Flexural Vibrations of Elastic Plates," *Zeitschrift fur Angewandte Mathematik und Mechanik (ZAMM)*, Dec. 1961, pp. 501–510.

3 Prathap, G., and Varadan, T. K., "On the Non-Linear Vibrations of Rectangular Plates," *J. of Sound and Vibration*, Vol. 56, No. 4, 1978, pp. 521–530. Also, letter to the editor, *J. of Sound and Vibration*, Vol. 61, No. 2, pp. 311–314.

4 Yamaki, N., Otomo, K., and Chiba, M., "Non-Linear Vibrations of a Clamped Circular Plate With Initial Deflection and Initial Edge Displacement, Part 1: Theory," *J. of Sound and Vibration*, Vol. 79, No. 1, 1981, pp. 23–42.

5 Chia, C. Y., *Nonlinear Analysis of Plates*," McGraw-Hill, New York, 1980.

6 Hui, D., and Leissa, A. W., "Effects of Geometric Imperfections on Vibrations of Biaxially Compressed Rectangular Flat Plates," ASME JOURNAL OF APPLIED MECHANICS, Vol. 51, 1984, pp. 00–00.

7 Hui, D., "Large Amplitude Axisymmetric Vibrations of Geometrically Imperfect Circular Plates," *J. of Sound and Vibration*, Vol. 91, No. 3, Dec. 1983, pp. 750–756.

8 Celep, Z., "Free Flexural Vibration of Initially Imperfect Thin Plates," *Zeitschrift fur Angewandte Mathematik und Mechanik (ZAMM)*, Vol. 56, Sept. 1976, pp. 423–428.

9 Hui, D., and Leissa, A. W., "Effects of Uni-Directional Geometric Imperfections on Vibrations of Pressurized Shallow Spherical Shells," *Int. J. of Nonlinear Mechanics*, Vol. 18, 1983, pp. 279–285.

10 Hui, D., "Large Amplitude Vibrations of Geometrically Imperfect Shallow Spherical Shells with Structural Damping," *AIAA Journal*, Vol. 21, 1983, pp. 1736–1741.

11 Hui, D., "Influence of Geometric Imperfections and In-Plane Constraints on Nonlinear Vibrations of Simply Supported Cylindrical Panels," ASME JOURNAL OF APPLIED MECHANICS, Vol. 51, 1984, in press.

12 Bert, C. W., "Material Damping: An Introductory Review of Mathematical Models, Measures and Experimental Techniques," *J. of Sound and Vibration*, Vol. 29, No. 2, 1973, pp. 129–153.

13 Tvergaard, V., "Influence of Post-Buckling Behaviour on Optimum Design of Stiffened Panels," *Int. J. of Solids and Structures*, Vol. 9, 1973, pp. 1519–1534.

14 Grossman, P. L., Koplik, B., and Yu, Y. Y., "Nonlinear Vibrations of Shallow Spherical Shells," ASME JOURNAL OF APPLIED MECHANICS, 1969, pp. 451–458.

15 Koiter, W. T., "On the Stability of Elastic Equilibrium," Doctoral Thesis, Delft, Amsterdam, 1945, English Translations: NASA TT-F10, 833, 1967 and AFFDL-TR-70-25, 1970.

16 Lau, S. L., and Cheung, Y. K., "Amplitude Incremental Variational Principle for Nonlinear Vibration of Elastic Systems," ASME JOURNAL OF APPLIED MECHANICS, Vol. 48, Dec. 1981, pp. 959–964.

17 Sathyamoorthy, M., "Effects of Large Amplitude, Shear and Rotatory Inertia on Vibration of Rectangular Plates," *J. of Sound and Vibration*, Vol. 63, 1979, pp. 161–167.