

# Amplitude modulation theory and its application to two-mode buckling problems \*)

By David Hui, The Ohio State University, Dept. of Engineering Mechanics, Columbus, Ohio 43210, U.S.A.

## 1. Introduction

Koiter's theory of amplitude modulation of the local mode is a very useful and elegant theory which is capable of predicting essential information such as the existence of the lower bound of the local buckling load and sensitivity to local and/or overall geometric imperfections. This theory was developed in the early 1970's on the basis of local short-wave mode modulation due to its interaction with the overall long-wave mode. According to this theory, the local mode exists only in the portion of the shell which bends inward. It was applied by Koiter and his associates to built-up columns [1], periodically stiffened flat plates [2, 3] and was generalized to stringer-reinforced cylindrical shells [4] with the possibility of further application to ring stiffened cylindrical shells. Moreover, applications can be made to any shell structures in which the two competing modes are one of long wave and one of short wave where both of these modes are periodic in at least one direction, such as stiffened spherical caps and stiffened conical shells. In these two-mode buckling problems, the amplitude modulation theory is generally superior to Koiter's general theory of elastic stability [5] due to its simplicity in computing the postbuckling coefficients and its validity for a larger range of imperfection amplitudes.

Several excellent in-depth investigations of interactive buckling of doubly symmetric plate structures and thin-walled columns were examined by Sridharan [6] Sridharan and Benito [7] and Benito and Sridharan [8]. However, apart from brief mention by Sridharan et al. [6–8], it appears that this powerful stability theory is still relatively obscure as it has not been applied to structural problems by people other than Koiter and his associates (with the exception of the author's Ph. D. thesis [9]).

The present paper is motivated by a review written by Byskov [10] on the author's earlier work [11] which mentioned that "It would have been interesting to see results for designs with local modes that are postbuckling unstable by

---

\*) Part of this work was presented at 15<sup>th</sup> ICTAM Conference, sponsored by IUTAM, August 1980.

themselves". Byskov's remark was based on his extensive experience in solving problems involving mode interaction of stiffened cylindrical shells under compression [12, 13]. Thus, in order to assess the effects of unstable local mode, an attempt is made to study the mode interaction problem of axially stiffened cylindrical shells under compression in which the local mode is unstable.

This article aims to review some of the basic features of Koiter's amplitude theory and to present it in a simple form suitable for general application to beam, plates and shells buckling problems. Primary emphasis will be placed on the local and overall two-mode interaction of stringer-reinforced cylindrical shells under compression. In particular, this theory is applied here to include the additional stabilizing effects which raise the quartic term of the potential energy of the local mode, such as stringer torsional rigidity and axial stiffness. A brief discussion relating this theory and Koiter's general theory of 1945 [5, 12–15] will be included. It is hoped that Koiter's modulation theory will become more familiar and accessible to researchers since most of the available literature is in the form of reports rather than in scientific journals.

## 2. Koiter's theory of amplitude modulation and its application

One of the oftenly encountered difficulties in multi-mode buckling of thin-walled structures is that there are a large number of buckling modes which have eigenvalues not much above the classical buckling load. These higher eigenvalues and the corresponding modes may have a considerable influence on the postbuckling problem, but are usually ignored (see Koiter's work on buckling of externally pressurized spherical shells [16]). As a result of two-mode interaction, the cubic term of the potential energy of the form  $\xi_1 \xi_2^2$  (where  $\xi_1$  and  $\xi_2$  are the amplitudes of the overall long-wave and local short-wave modes, respectively) is no longer zero so that the computation of the quartic term of the form  $\xi_1^2 \xi_2^2$  is not necessary. Due to mode interaction, the amplitude of the local panel mode is no longer a constant, but is now a function of the axial and circumferential coordinates  $X$  and  $Y$  respectively. For stringer-reinforced cylindrical shells, the local deflection mode can be written as ( $\nu$  is Poisson's ratio and  $R$  is the radius)

$$W_2(X, Y) = (t) h(x, y) \sin(\lambda \pi X/d_s) w_c(y) \quad (1)$$

$$(x, y) = (q_0/R)(X, Y), \quad q_0 = (2cR/t)^{1/2}, \quad c = [3(1 - \nu^2)]^{1/2}$$

where  $d_s$  is the curved distance between adjacent equally spaced stringers,  $t$  is the skin thickness and  $w_c(y)$  is the local mode described by Stephens [17]. Note that the symbol  $h(x, y)$  is used here instead of  $f(x, y)$  in [4] in order to avoid confusion with the stress function. On the other hand, the overall mode shape remains exactly the same as the one presented by Hutchinson and Amazigo [18]. In the case of classical simple support at the two ends  $X = 0, L$  ( $W = 0, W_{,XX} = 0,$

$V = 0$  and  $N_x = 0$ ), the overall mode is ( $m_1$  and  $n_1$  are positive integers and  $E$  is Young's modulus of the skin),

$$[W_1(X, Y), F_1(X, Y)] = (\xi_1) (t, \bar{e} E t^3) \sin(m_1 \pi X/L) \sin(n_1 Y/R). \tag{2}$$

Within the assumptions employed in [4], the two-mode initial postbuckling problem of a stringer-reinforced cylindrical shell is solved provided the following non-dimensional parameters are computed. They are the coefficient of the transformed quartic term of the potential energy of the overall mode  $C_1$ , the  $b$  coefficient (see [9]) of the local mode  $b_2$ , the coefficient of the cubic interaction term  $B$  and the ratio of the overall to the local classical buckling load  $r$  (defined to be  $\sigma_c(\text{overall})/\sigma_c(\text{local})$ ). The remainder of this section is devoted to the derivation of the energy expression as a function of these four parameters.

The quadratic term of the potential energy which depends on the applied load (with unit force-length and  $F_p$  is the pre-buckling stress function),

$$P'_2[u] = (1/2) \int_0^{2\pi R} \int_0^L (W, x)^2 F_{p, YY} dX dY \tag{3}$$

where  $F_p = F_{p1} + F_{p2}$ . For the overall mode,

$$F_{p1, YY} = P/(-2\pi R) = (-\bar{\sigma})(t)(1 + \alpha_s) \tag{4}$$

where  $P$  is the total force on the skin as well as on the stringers and  $\bar{\sigma}$  is the applied stress (the stress on the stringer is the same as the stress on the skin) and  $\alpha_s = M_s A_s / (2\pi R) = A_s / (d_s t) =$  ratio of stringer area to the panel area. Thus, the Taylor series expansion of the above energy expression for the overall mode is, ( $\bar{\sigma}_1$  is the overall buckling stress and  $W_1$  is the overall buckling deflection mode),

$$(\bar{\sigma} - \bar{\sigma}_1) \frac{\partial}{\partial \bar{\sigma}} (P'_2[u_1]) = (\bar{\sigma}_1) [1 - (\bar{\sigma}/\bar{\sigma}_1)] (1 + \alpha_s) (t/2) \int_0^{2\pi R} \int_0^L (W_{1, x})^2 dX dY. \tag{5}$$

Introducing the non-dimensional applied load  $\sigma = Rc \bar{\sigma} / (Rt)$  and carrying out the above integrations, the result is ( $\bar{\sigma}/\bar{\sigma}_1 = \sigma/\sigma_1$ )

$$(\bar{\sigma} - \bar{\sigma}_1) \frac{\partial}{\partial \bar{\sigma}} (P'_2[u_1]) = [1 - (\sigma/\sigma_1)] (Z_1 \xi_1^2) (Et) (2\pi RL) \tag{6}$$

$$Z_1 = [\sigma_1 / (8c)] (1 + \alpha_s) (m_1 \pi)^2 (t/R)^3 (R/L)^2.$$

In a similar manner, the Taylor series expansion of the quadratic term for the "modulated" local mode is ( $\bar{\sigma}_2$  is the local buckling stress),

$$(\bar{\sigma} - \bar{\sigma}_2) \frac{\partial}{\partial \bar{\sigma}} (P'_2[u_2]) = (\bar{\sigma}_2) [1 - (\sigma/\sigma_2)] (t/2) \int_0^{2\pi R} \int_0^L (W_{2, x})^2 dX dY. \tag{7}$$

Note that the factor  $(1 + \alpha_s)$  does not appear in the above expression because  $F_{p2, yy}$  is defined to be,

$$F_{p2, YY} = (P - P_{st}) / (-2\pi R) = P_{sk} / (2\pi R) = \bar{\sigma} t \tag{8}$$

where  $P_{st}$  is the force on all the stringers and  $P_{sk}$  is the force on the skin from  $Y = 0$  to  $Y = 2\pi R$  (note that  $\bar{\sigma} = P_{sk}/(2\pi R t) = P/[2\pi R t(1 + \alpha_s)]$ ). Furthermore, the  $X$  derivative of the modulated local mode is,

$$W_{2,x} = (t) h(x, y) (\lambda \pi/d_s) \cos(\lambda \pi X/d_s) w_c(y) \\ + (t) h(x, y),_x (\lambda \pi/d_s) \sin(\lambda \pi X/d_s) w_c(y). \quad (9)$$

Within the assumptions of the modulation theory,  $h(x, y)$  is a slowly varying function of the  $x$  and  $y$  coordinates. Thus, the second term (which involves  $h(x, y),_x$ ) may be neglected. Moreover, the following integration simplification is permitted,

$$\iint \left( \begin{array}{c} \text{short-wave} \\ \text{quantities} \end{array} \right) \left( \begin{array}{c} \text{long-wave} \\ \text{quantities} \end{array} \right) dX dY = \iint \left( \begin{array}{c} \text{short-wave} \\ \text{quantities} \end{array} \right) dX dY \\ \cdot \iint \left( \begin{array}{c} \text{long-wave} \\ \text{quantities} \end{array} \right) dX dY. \quad (10)$$

Thus, the expanded quadratic term for the modulated local mode becomes,

$$(\bar{\sigma} - \bar{\sigma}_2) \frac{\partial}{\partial \bar{\sigma}} (P'_2[u_2]) = (Et) [1 - (\sigma/\sigma_2)] Z_2 \int_0^{2\pi R} \int_0^L h(x, y)^2 dX dY \quad (11)$$

where (the flatness parameter  $\theta$  is defined to be  $q_0/M_s$  and  $M_s = 2\pi R/d_s$  is the number of stringers),

$$Z_2 = (\sigma_2) [t/(Rc)] (t/d_s)^2 (\lambda \pi)^2 (H_1/4) = (\sigma_2) [H_1/(16c)] (t/R)^3 (\lambda M_s)^2 \\ H_1 = [1/(\pi\theta)] \int_0^{\pi\theta} [w_c(y)]^2 dy. \quad (12)$$

The cubic term of the potential energy of a cylindrical shell is

$$P_3[u] = (1/2) \int_0^{2\pi R} \int_0^L \{F_{,YY} (W_{,X})^2 + F_{,XX} (W_{,Y})^2 \\ - 2F_{,XY} (W_{,X})(W_{,Y})\} dX dY. \quad (13)$$

Further, for the present two-mode buckling problem,  $W = W_1 + W_2$  and  $F = F_1 + F_2$ . Due to the sinusoidal properties of the two modes,

$$P_3[u_1] = 0, \quad P_3[u_2] = 0, \quad P_{21}[u_1, u_2] = 0 \quad (14)$$

and the only non-zero cubic term is (retaining only the predominating terms),

$$P_{12}[u_1, u_2] = (1/2) \int_0^{2\pi R} \int_0^L \{N_x(W_{2,X})^2 + N_y(W_{2,Y})^2\} dX dY. \quad (15)$$

It should be noted that the skin stress resultants  $N_x$  and  $N_y$  are not to be confused with the „smeared-out“ quantities  $F_{1,YY}$  and  $F_{1,XX}$  as these latter two quantities

refer to the smeared-out stress resultants. From Hutchinson-Amazigo's work [18], the skin stress resultants  $N_x$  and  $N_y$  are related to the smeared-out quantities  $F_{1,YY}$  and  $F_{1,XX}$  by,

$$\begin{aligned} N_x &= (A_{xx}F_{1,XX} + A_{xy}F_{1,YY}) + (\bar{B}_{xx}W_{1,XX} + \bar{B}_{xy}W_{1,YY})(Et^2) \\ N_y &= (A_{yy}F_{1,YY} + A_{yx}F_{1,XX}) + (\bar{B}_{yy}W_{1,YY} + \bar{B}_{yx}W_{1,XX})(Et^2) \end{aligned} \tag{16}$$

where the non-dimensional quantities  $A_{xx}, A_{xy}, A_{yy}, A_{yx}, \bar{B}_{xx}, \bar{B}_{xy}, \bar{B}_{yy}$  and  $\bar{B}_{yx}$  can be obtained in [18]. Thus, in the case of classical simply supported stringer-reinforced cylindrical shell, one obtains ( $S_x$  and  $S_y$  are defined later),

$$\begin{aligned} N_x &= (Et)(\xi_1 S_x) \sin(m_1 \pi X/L) \sin(n_1 Y/R) \\ N_y &= (Et)(\xi_2 S_y) \sin(m_1 \pi X/L) \sin(n_1 Y/R). \end{aligned} \tag{17}$$

Thus, the desired expression for the non-zero cubic interaction term is,

$$\begin{aligned} P_{12}[u_1, u_2] &= (\xi_1/2) \int_0^{2\pi R} \int_0^L \{(2)^{1/2} \sin(m_1 \pi X/L) \sin(n_1 Y/R) h(x, y)^2\} dX dY \\ &\cdot \left( \frac{(Et^3/\sqrt{2})}{(2\pi q_0^2(L/R))} \right) \int_0^{2\pi R} \int_0^L \{S_x(\lambda\pi/d_s)^2 \cos^2(\lambda\pi X/d_s) [w_c(y)]^2 \\ &+ S_y \sin^2(\lambda\pi X/d_s) [w_c(y), y]^2\} dX dY. \end{aligned} \tag{18}$$

It can be simplified to become,

$$\begin{aligned} P_{12}[u_1, u_2] &= (Et)(\xi_1 Z_3) \int_0^{2\pi R} \int_0^L \\ &\cdot \{(2)^{1/2} \sin(m_1 \pi X/L) \sin(n_1 Y/R) h(x, y)\} dX dY \\ Z_3 &= (t/R)^2 [M_s^2/(4\sqrt{2})] [S_x \lambda^2 (H_1/4) + S_y \theta^2 H_2] \end{aligned} \tag{19}$$

and  $H_2, S_x$  and  $S_y$  are defined to be,

$$\begin{aligned} H_2 &= [1/(\pi\theta)] \int_0^{\pi\theta} [w_c(y), y]^2 dx dy \\ S_x &= [-A_{xx}(m\pi)^2(t/L)^2 - A_{xy}n^2(t/R)^2](\bar{e}) - [\bar{B}_{xx}(m\pi)^2(t/L)^2 \\ &+ \bar{B}_{xy}n^2(t/R)^2] \\ S_y &= [-A_{yy}n^2(t/R)^2 - A_{yx}(m\pi)^2(t/L)^2](\bar{e}) - [\bar{B}_{yy}n^2(t/R)^2 \\ &+ \bar{B}_{yx}(m\pi)^2(t/L)^2]. \end{aligned} \tag{20}$$

Assembling the various terms, the potential energy of the two-mode interaction problem within the framework of Koiter's modulation theory is,

$$\begin{aligned}
 \frac{\text{P. E.}}{(E t)(2 \pi R L)} &= (Z_1/2) (b_1) \xi_1^4 + (Z_2/2) (b_2) \int_0^{2 \pi R} \int_0^L h(x, y)^4 dS \\
 &+ (Z_3 \bar{\xi}_1) \int_0^{2 \pi R} \int_0^L \{(2)^{1/2} \sin(m_1 \pi X/L) \sin(n_1 Y/R) h(x, y)^2\} dS \\
 &+ [1 - (\sigma/\sigma_1)] Z_1 \xi_1^2 + [1 - (\sigma/\sigma_2)] Z_2 \int_0^{2 \pi R} \int_0^L h(x, y)^2 dS \\
 &- 2 Z_1(\sigma/\sigma_1) (\bar{\xi}_1 \xi_1) - 2 Z_2(\sigma/\sigma_2) \int_0^{2 \pi R} \int_0^L \bar{h}(x, y) h(x, y) dS \tag{21}
 \end{aligned}$$

where  $dS = (dX dY)/(2 \pi R L)$ , the barred quantities denote imperfections and  $b_1$  and  $b_2$  are the  $b$  coefficients of the overall and local modes respectively.

In order to arrange the above potential energy expression in the form of Ref. 4, the following transformation is introduced,

$$\begin{aligned}
 (a_1, \bar{a}_1) &= (\xi_1, \bar{\xi}_1) (Z_1/Z_2)^{1/2}, \quad B = Z_3/[2 (Z_1 Z_2)^{1/2}] \\
 C_1 &= (Z_2/Z_1) b_1, \quad \lambda^* = \sigma/\sigma_2, \quad r = \sigma_1/\sigma_2, \quad \sigma/\sigma_1 = \lambda^*/r \tag{22}
 \end{aligned}$$

so that the final form of the two-mode potential energy expression becomes,

$$\begin{aligned}
 \frac{\text{P. E.}}{(E t)(2 \pi R L)} &= [1 - (\lambda^*/r)] (a_1)^2 + (1 - \lambda^*) \iint h(x, y)^2 dS \\
 &+ (C_1/2) (a_1)^4 + (b_2/2) \iint h(x, y)^4 dS \\
 &+ (2 B a_1) \iint (2)^{1/2} \sin(m_1 \pi X/L) \sin(n_1 Y/R) h(x, y)^2 dS \\
 &- 2(\lambda^*/r) (\bar{a}_1 a_1) - (2 \lambda^*) \iint \bar{h}(x, y) h(x, y) dS. \tag{23}
 \end{aligned}$$

Thus, the present overall and local mode interaction problem is characterized by only four non-dimensional parameters: the transformed coefficient of the quartic term  $C_1$ , the  $b$  coefficient of the local mode  $b_2$ , the coefficient of the cubic interaction term  $B$  and the ratio of the overall buckling load to the local buckling load  $r$ . Minimizing the above potential energy with respect to the amplitudes  $a_1$  and  $h(x, y)$ , one obtains two equilibrium equations:

$$\begin{aligned}
 [1 - (\lambda^*/r)] (a_1) + C_1 (a_1)^3 + B \iint 2^{1/2} \sin(m_1 \pi X/L) \sin(n_1 Y/R) h(x, y)^2 dS \\
 = (\lambda^*/r) (\bar{a}_1) \tag{24}
 \end{aligned}$$

$$\begin{aligned}
 \iint \{(1 - \lambda^*) h(x, y) + b_2 h(x, y)^3 + (2 B a_1) (2)^{1/2} \sin(m_1 \pi X/L) \\
 \cdot \sin(n_1 Y/R) h(x, y) - 2 \lambda^* \bar{h}(x, y)\} dS = 0. \tag{25}
 \end{aligned}$$

Finally, in the presence of a local imperfection alone ( $a_1 = 0$ ), the local buckling load of the imperfect system  $\lambda_c^*$  is related to the amplitude of the local

imperfection  $h_0$  by

$$(1 - \lambda_c^*) \left( \frac{2B^2 - 2b_2[1 - (\lambda_c^*/r)]}{2B^2 - 3b_2[1 - (\lambda_c^*/r)]} \right) \left( \frac{(1 - \lambda_c^*)[1 - (\lambda_c^*/r)]}{2B^2 - 3b_2[1 - (\lambda_c^*/r)]} \right)^{1/2} = (\lambda_c^*) h_0. \quad (26)$$

Further, in the event that the  $b$  coefficient of the local panel mode  $b_2$  turns out to be positive, the above relation yields a lower bound [4],

$$\text{Lower Bound} = (r) [1 - (2B^2/(3b_2))]. \quad (27)$$

This lower bound means that the shell structure cannot buckle below this value regardless of how large the amplitude of the local panel geometric imperfection (of course, it has to be within the assumptions of the modulation theory). It should be noted that the buckling load  $\lambda_c^*$  and the lower bound are quite sensitive to the  $b$  coefficient of the local mode  $b_2$ . Since it has been demonstrated [9] that the stringer axial stiffness and torsional rigidity ratio play an important role in affecting the  $b_2$  coefficient, it is obvious that they will play an equally important role in the above two-mode interaction problem using Koiter's modulation theory.

At this stage, it is of interest to write down the two equilibrium equations for the two-mode interaction problem using Koiter's general theory of 1945 [5]. Retaining only the predominant terms for axially stiffened cylindrical shells, one obtains (Byсков and Hutchinson [12] and Hui [15]),

$$\begin{aligned} b_1 \xi_1^3 + b_{12} \xi_1 \xi_2^2 + [1 - (\sigma/\sigma_1)] \xi_1 &= (\sigma/\sigma_1) \bar{\xi}_1 \\ b_2 \xi_2^3 + b_{21} \xi_1^2 \xi_2 + [1 - (\sigma/\sigma_2)] \xi_2 &= (\sigma/\sigma_2) \bar{\xi}_2. \end{aligned} \quad (28)$$

It should be mentioned that these equilibrium equations are valid only for sufficiently small values of the imperfection amplitudes. In general, the computation of the  $b_{12}$  coefficient requires considerably more analytical work than the computation of the  $B$  coefficient in Koiter's amplitude modulation theory. The range of validity for the imperfection amplitudes using Koiter's modulation theory is larger than the range using Koiter's 1945 theory. Note that the equilibrium equations obtained using Koiter's modulation theory (Equations 24 and 25) may be grossly inaccurate if the  $B$  coefficient turns out to be extremely small so that the higher order quartic terms of the two-mode problem are no longer negligible for practical values of the imperfection amplitude. It is not clear whether these two stability theories are asymptotically equivalent (that is, for vanishingly small imperfection amplitudes). However, one can ascertain that Koiter's general theory of 1945 is asymptotically exact and the accuracy decreases with increasing imperfection amplitudes. On the contrary, Koiter's modulation theory is "probably" quite close to being asymptotically exact, but the accuracy decreases with increasing imperfection amplitude to a much smaller extent.

### 3. Example problem

As an example problem, an outside stringer (rectangular shape) reinforced cylindrical shell with structural parameters which are identical to those considered by Koiter [4] is examined. Following Koiter's method of choosing the optimum design [4] (Koiter kept the stringer eccentricity fixed and allow the thickness of the stringer to vary with the number of stringers; this method is quite different from Byskov and Hutchinson [12] who kept the thickness of the stringer fixed and allow the eccentricity to vary with the number of stringers), the fixed parameters are,

$$\alpha_s = 0.5, \quad \nu = 0.3, \quad e_s/t = 3.17, \quad E_n = E_s/E = 1.0, \quad R/t = 420, \quad R/L = 2 \quad (29)$$

where  $\nu$  is Poisson's ratio,  $e_s$  is stringer eccentricity and  $E_s$  is stringer Young's modulus. The above data imply that the following parameters are also kept fixed,

$$\begin{aligned} \beta_s = E_s I_s / D d_s = 12.974, \quad q_0 = 37.2546, \quad Q_s/t = 2(e_s/t) - 1 = 5.34 \\ Z = [L^2 / (Rt)] (1 - \nu^2)^{1/2} = 100.1636, \quad c = 1.65227, \quad L/R = 0.5 \end{aligned} \quad (30)$$

where  $I_s$  is the stringer moment of inertia as defined in [9, 18],  $D$  is the skin flexural rigidity. The shell parameters which change as the number of stringers  $M_s$  varies are  $\theta$ ,  $d_s/t$ , the torsional rigidity  $\gamma_s$  as defined in [17] and the tangential bending stiffness ratio  $\beta_t = E_s I_t / (D d_s)$ . Further, the number of local axial half sine-waves  $m_2$  is related to the non-dimensional wave number  $\lambda$  as defined in [9] by  $m_2 = \lambda M_s / (4\pi)$ . This method of parameter variation has the virtue of the overall buckling load and overall  $b$  coefficient (within the assumptions of the smeared-out theory) remaining fixed for any chosen design.

Figure 1 shows a picture of an axially stiffened cylindrical shell under compression in which both local buckles and overall mode (indicates by the bending of the stringers) occur simultaneously. Some of these experimental results were presented at the IUTAM symposium (Tennyson, Booton and Hui [19]).

The classical buckling loads for the local and overall modes (the overall mode is assumed to satisfy the classical simple support condition) are plotted in Figure 2 using the above shell parameters. Since the aim of the present investigation is to show the importance of stringer axial stiffness, the effect due to the discreteness of the local wavelength parameter (which may be fairly significant since the shell is relatively short) is neglected. The present buckling load is almost identical to that obtained by Stephens and it can be seen that it is significantly higher than Koiter's 1956 result [20]. The difference in the buckling loads is due to the torsional rigidity of the stringers. Consequently, the optimum number of

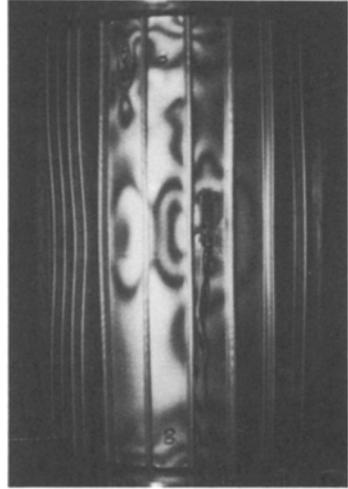


Figure 1  
 Simultaneous Local and Overall Buckling of an Integrally Stiffened Cylindrical Shell under Axial Compression.

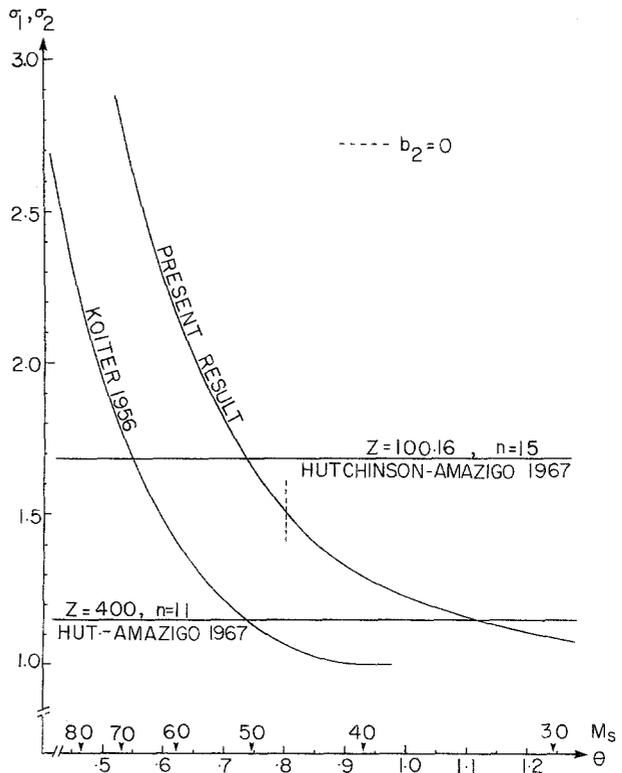


Figure 2  
 Local and Overall Buckling Loads versus Flatness Parameter.

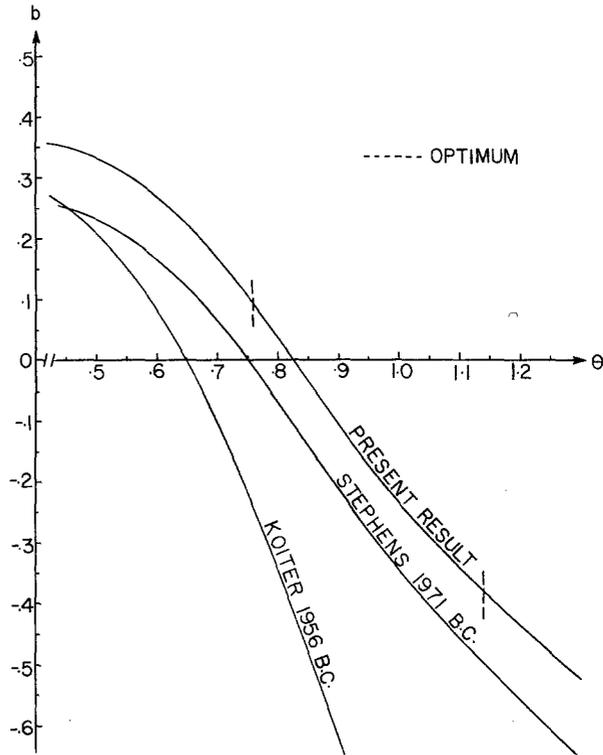


Figure 3  
The  $b$  Coefficient of the Local Mode versus the Flatness Parameter.

stringers for simultaneous local and overall buckling occurs at

$$\begin{aligned}
 \theta &= 0.760 \text{ with } M_s = 49 \text{ at } \sigma = 1.682 \text{ present optimum} \\
 \theta &= 0.571 \text{ with } M_s = 65 \text{ again at } \sigma = 1.682 \text{ optimum using Koiter's (31)} \\
 &\qquad\qquad\qquad 1956 \text{ B.C.}
 \end{aligned}$$

The corresponding  $b$  coefficient, where it is assumed that the wave number  $\lambda$  is continuous, is plotted in Figure 3. Again, it appears that there is a roughly constant upward shift since the area ratio  $\alpha_s$  is fixed to be 0.5 for all designs. At the present optimum design ( $\theta = 0.760$ ), the  $b$  coefficient is 0.1 and thus, the local mode alone is postbuckling stable. Moreover, the sign change in the  $b$  coefficient occurs at,

$$\begin{aligned}
 \theta &= 0.825 \text{ with } M_s = 45 \text{ present result} \\
 \theta &= 0.705 \text{ with } M_s = 53 \text{ Stephens' 1971 B.C.} \\
 \theta &= 0.648 \text{ with } M_s = 57 \text{ Koiter's 1956 B.C.}
 \end{aligned} \tag{32}$$

Thus, the present result predicts stable local postbuckling for a much wider range of shell parameter variations than that predicted by using Stephens' or Koiter's boundary conditions.

In view of the fact that there are only about six axial half waves in the local mode near the optimum design and that the results presented in Figure 3 are strictly valid only if the shell is sufficiently long, it is of interest to investigate a longer cylindrical shell ( $Z = 400$ ,  $R/L = 1.001636$ ) while keeping the same cross-section. Doing so, the optimum design for simultaneous local and overall buckling occurs at,

$$\begin{aligned} \theta &= 1.14 \quad \text{with } M_s = 33 \quad \text{at } \sigma = 1.15 \quad \text{present result} \\ \theta &= 0.755 \quad \text{with } M_s = 49 \quad \text{again at } \sigma = 1.15 \quad \text{Koiter's 1956 B.C.} \end{aligned} \quad (33)$$

Moreover, since the local wavelength parameter  $\lambda$  is assumed to be continuous, the  $b$  coefficient in Figure 3 remains valid for  $Z = 400$  as it was in the shorter shell  $Z = 100.16$ . It can be seen that the  $b$  coefficient of the actual optimum is  $-0.41$  and thus, it is postbuckling unstable.

Finally, the local and overall mode interaction problem is solved using Koiter's theory of amplitude modulation of the local mode. The cylindrical shell parameters along with the four non-dimensional quantities necessary to specify the two-mode potential energy expression ( $C_1$ ,  $b_2$ ,  $B$  and  $r$ ) are computed and tabulated in Table 1 for the two actual optima being considered:

$$\begin{aligned} Z = 100.16, \quad \theta = 0.7451 \quad \text{with } M_s = 50 \\ Z = 400, \quad \theta = 1.164 \quad \text{with } M_s = 32. \end{aligned} \quad (34)$$

Of particular interest is the bifurcation load from the local pre-buckling state to the overall mode. Note that the initial postbuckling state of the local mode (where the stringers remain more or less straight with respect to out-of-plane

Table 1  
Summary of the mode interaction parameters for Axially Stiffened Cylindrical Shells ( $r$  and  $B$  are the same for both the present and Stephens analyses).

$Z$	100.16	400
$\theta$	0.745093	1.16421
$M_s$	50	32
$r$	0.9685	1.018
$B$	0.1425	0.1206
$b_2$ (present)	0.11	-0.41
$b_2$ (Stephens)	0.015	-0.50
Lower Bound (present)	(0.8768) $r$ = 0.8493	no L.B.
Lower Bound (Stephens)	(0.0971) $r$ = 0.094	no L.B.
$a_1$	0.22 $\xi_1$	0.2655 $\xi_1$
$C_1$	-3.097	-0.861

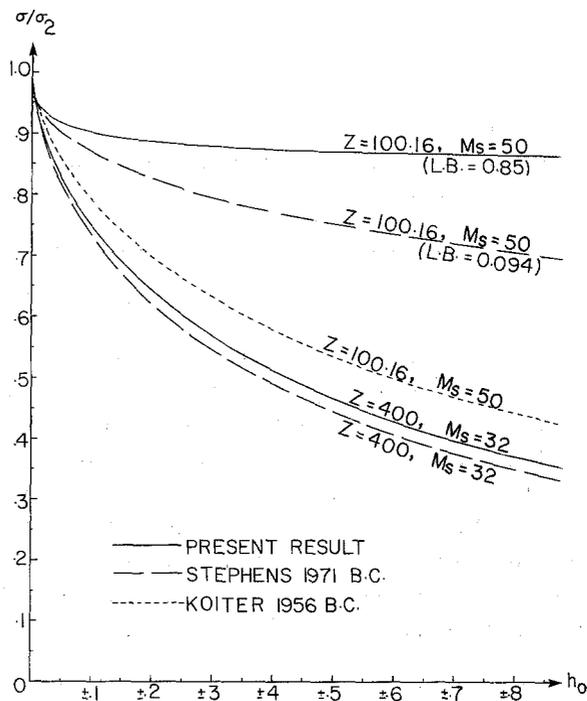


Figure 4  
Overall Bifurcation Load versus a Purely Local Imperfection Amplitude using Amplitude Modulation Theory.

bending) serves as the pre-buckling state for the overall mode, causing a nonlinear pre-buckling state. The bifurcation load (again, defined to be  $\sigma/\sigma_2$  where it should be cautioned that  $\sigma_2$  is different for the present and Koiter's 1956 result) is plotted against local imperfection amplitude normalized with respect to the skin thickness in Figure 4. It can be seen that the present critical load versus local imperfection curve levels off quite rapidly for imperfection amplitudes likely to occur in practice and the lower bound is much higher than that obtained using Stephens' 1971 or Koiter's 1956 boundary conditions. The Koiter 1956 curve for the longer stiffened cylindrical shell ( $Z = 400$ ) is not applicable since the  $b$  coefficient of the local mode is not valid for  $\theta \geq 1.0$  (that is, Koiter's 1956 analysis, which neglected the torsional rigidity of the stringers, allowed single-mode local panel analysis provided  $\theta \leq 1.0$ ). The optimum design for the longer shell ( $Z = 400$ ) at which the  $b$  coefficient of the local mode is "negative" is of interest as it demonstrates a more severe degrading effects of mode-interaction in reducing the load carrying capacity of the structure.

Finally, the above results should best be viewed qualitatively since each overall circumferential half-wave involves less than two stringers so that the smeared-out overall mode analysis may involve some minor inaccuracies. Nevertheless, this example problem is of interest since it was discussed by both Koiter [4] and Byskov-Hutchinson [12].

#### 4. Concluding remarks

Koiter's theory of amplitude modulation of the local mode has been reviewed and demonstrated in a practical example problem which involves a study of local and overall mode interaction of axially stiffened cylindrical shells. Of particular interest, it has been found that there is a significant "positive shift" of the  $b$  coefficient of the local mode due to the inclusion of the axial stiffness of the stringers, in addition to the stabilized influence of stringer torsional rigidity. The local and overall mode interaction problem is studied using Koiter's theory of amplitude modulation, using the presently derived  $b$  coefficient of the local mode. As expected, there is a significant "leveling off" of the critical load versus a purely local imperfection amplitude curves. This implies that the mode interaction problem between these two modes is less serious than it was thought to be, at least for the shell parameters being considered. Extension of the present work to mode-interaction of laminated stringer-reinforced cylindrical shells under compression [21, 22] is in progress.

#### Acknowledgement

The author appreciates valuable discussions with Professor W. T. Koiter on various aspects of mode interaction related to the title problem. The work was supported by the National Science and Engineering Research Council of Canada under Grant A-2783 through Professor R. C. Tennyson (who served as the thesis advisor of the author). Partial financial support was provided by The Ohio State University Seed Grant through the Office of Research and Graduate Studies.

#### References

- [1] W. T. Koiter, and G. D. C. Kuiken, *The interaction between local buckling and overall buckling on the behaviour of built-up columns*. Delft University of Technology, Report WTHD 23, May 1971.
- [2] W.T. Koiter and M. Pignataro, *An alternative approach to the interaction between local and overall buckling in stiffened panels*. Buckling of structures, IUTAM Symp., Harvard University, 1974. Edited by B. Budiansky, Springer-Verlag, Berlin 1976, pp. 133–148.
- [3] W. T. Koiter and M. Pignataro, *A general theory for the interaction between local and overall buckling of stiffened panels*. Delft University of Technology, Report WTHD 83, April 1976; also Estratto dal volume, *Problemi Attuali di Meccanica Teorica e Applicata*, Atti del Convegno internazionale a ricordo di Modesto Panetti, Torino 1977, pp. 179–222.
- [4] W. T. Koiter, *General theory of mode interaction in stiffened plate and shell structures*. Delft University of Technology, Report WTHD 91, September 1976.
- [5] W. T. Koiter, *On the stability of elastic equilibrium*. Doct Thesis, Delft University, Amsterdam, 1945; English Translations: NASA-TT-F10-833, 1967 and AFFDL-TR-70-25, 1970.
- [6] S. Sridharan, *Doubly symmetric interactive buckling of plate structures*. Int J Solids & Struct, 19, 625–641 1983.
- [7] S. Sridharan and R. Benito, *Columns: Static and dynamic interactive buckling*. J Eng Mech, ASCE, 110, 49–65, January 1984.

- [8] R. Benito and S. Sridharan, *Interactive buckling analysis with finite strips*. Int. J. Num. Meth. Eng., 1984, accepted for publication.
- [9] D. Hui, *Interaction between local and overall buckling modes in axially stiffened cylindrical shells*. University of Toronto, Institute for Aerospace Studies, Ph. D. thesis, 1981.
- [10] E. Byskov, *A review of the paper: Mode interaction of axially stiffened cylindrical shells; Effects of Stringer Axial Stiffness, Torsional Rigidity, and Eccentricity*. Applied Mechanics Review, ASME, 35, No. 8, August 1982, Review # 8271, p. 1224.
- [11] D. Hui, R. C. Tennyson, and J. S. Hansen, *Mode interaction of axially stiffened cylindrical shells: Effects of stringer axial stiffness, torsional rigidity and eccentricity*. ASME J Appl Mech, 48, 915–922, Dec 1981.
- [12] E. Byskov and J. W. Hutchinson, *Mode interaction in axially stiffened cylindrical shells*. AIAA J, 15, 941–948, July 1977.
- [13] E. Byskov and J. C. Hansen, *Postbuckling and imperfection sensitivity analysis of axially stiffened cylindrical shells with mode interaction*. J Struct Mech, 8, 205–224, 1980, also DCAMM Report 163, August 1979.
- [14] D. Hui and J. S. Hansen, *Two-mode buckling of an elastically supported plate and its relation to catastrophe theory*. ASME J Appl Mech, 47, 607–612, September 1980.
- [15] D. Hui, *Effects of mode interaction on collapse of short, imperfect thin-walled columns*. ASME J Appl Mech 50, 566–573, September 1984.
- [16] W. T. Koiter, *The non-linear buckling problem of a complete spherical shell under uniform external pressure*. Proc Koninklijke Nederlandse Akademie Van Wetenschappen, Ser. B, 72, 40–123 1969.
- [17] W. B. Stephens, *Imperfection-sensitivity of axially compressed stringer reinforced cylindrical panels under internal pressure*. AIAA J, 9, 1713–1719, September 1971.
- [18] J. W. Hutchinson and J. C. Amazigo, *Imperfection-sensitivity of eccentrically stiffened cylindrical shells*. AIAA J, 5, 392–401, March 1967.
- [19] R. C. Tennyson, M. Booton, and D. Hui, *Changes in cylindrical shell buckling behaviour resulting from system parameter variations*. Proc. 15th Int Congress of Applied Mechanics, edited by Rimrott, F. P. J. and Tabarrok, B., sponsored by IUTAM, pp. 417–428, August 1980.
- [20] W. T. Koiter, *Buckling and post-buckling behaviour of a cylindrical panel under axial compression*. National Aeronautical Research Inst, Amsterdam, Report S-476, May 1956.
- [21] R. C. Tennyson and J. S. Hansen, *Optimum design for buckling of laminated cylinders*. Collapse: The buckling of structures in theory and practice, Proc. of IUTAM Symp, edited by Thompson, J. M. T. and Hunt, G. W., August 1982, Cambridge University Press, 1983, pp. 409–429.
- [22] D. Hui, *Asymmetric postbuckling of symmetrically laminated cross ply, short cylindrical panels under compression*. Composite Structures, 3, 81–95, 1985.

### Abstract

This paper aims to review Koiter's theory of amplitude modulation of the local mode and present it in a form suitable for general application. Various features of this theory are compared with Koiter's general theory of 1945. The amplitude modulation theory is applied to two-mode buckling of stringer stiffened cylindrical shells under axial compression. New mode interaction results are reported involving the simultaneous local and overall buckling in which the local mode is postbuckling unstable.

### Zusammenfassung

Die vorliegende Arbeit setzt sich zum Ziel, die Koitersche Theorie der lokalen Amplitudenmodulation in einer Form zu behandeln welche sich leichter verallgemeinern läßt. Verschiedene Eigenschaften dieser Theorie werden mit der allgemeinen Koiterschen Theorie von 1945 verglichen. Die Theorie der Amplitudenmodulation wird auf das Beulproblem in zwei Moden von versteiften zylindrischen Schalen unter axialer Kompression angewendet. Neue Resultate der modalen Interaktion lokalen und globalen Beulens mit lokaler Instabilität im Nachbeulzustand werden aufgeführt.

(Received: April 4, 1985)