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SOFT-SPRING NONLINEAR VIBRATIONS OF ANTISYMMETRICALLY LAMINATED RECTANGULAR PLATES*

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Summary—This paper deals with the effects of initial geometric imperfections and in-plane boundary conditions on the large-amplitude vibration behavior of angle- and cross-ply rectangular thin plates. It is found that the presence of imperfection amplitudes of the order of only half the total laminated-plate thickness may significantly raise the vibration frequencies and change the large-amplitude vibration behavior from the well-known hard-spring to soft-spring behavior. The effects of fibre angles and bending–stretching coupling for angle-ply plates and Young's moduli ratios and number of layers for antisymmetric cross-ply plates are examined.

1. INTRODUCTION

Large-amplitude vibrations of composite plates has been a topic which has attracted considerable attention over the past 15 years. This topic has become increasingly important today due to its widespread applications in mechanical, aerospace and ocean-engineering structural configurations. Subsequent to the early papers on the effects of bending–stretching coupling of composite plates (Reissner and Stavsky [1]; Ambartsumyan [2]), large amplitude vibrations of anti-symmetric angle-ply plates were first investigated by Bennett [3]. The analysis was further examined by Bert [4] and Chandra and Basava Raju [5]. Large-amplitude vibrations of anti-symmetric cross-ply plates were analysed by Chandra *et al.* [6, 7]. Moreover, the effects of shear and rotatory inertia on large amplitude vibrations of plates were examined by Wu and Vinson [8], Sathyamoorthy and Chia [9–10] and Celep [11] and a comparison of new composite-plate theories was presented by Bert [12]. Further results using the finite-element method were obtained by Reddy and Chao [13]. The above papers are by no means exhaustive and excellent reviews on these problems were written by Chia [14], Bert [15] and Leissa [16].

On the other hand, attempts were also made to investigate the effects of geometric imperfections on the linear and nonlinear vibration behavior of isotropic homogeneous rectangular plates (Celep [11], Yamaki and Chiba [17] and Hui [18]). Further research on the effects of geometric imperfections on the large-amplitude vibrations of simply supported and clamped circular plates [19], simply supported cylindrical panels with in-plane constraints [20], almost simply supported circular cylindrical shells [21] and shallow spherical shells [22] were examined. All the above investigations deal with isotropic homogeneous materials. It appears that the present paper is the first attempt to study the effects of geometric imperfections on large-amplitude vibrations of composite plates which may exhibit bending–stretching coupling. Further studies on their effects on composite cylindrical shells or cylindrical panels will be presented in separate papers.

The present paper aims to study the effects of geometric imperfections and in-plane constraints on the linear as well as nonlinear vibration behavior of clamped angle- and cross-ply rectangular thin plates. The analysis is based on a solution of the dynamic analogue of von Kármán differential equations valid for moderately large deflections. The imperfection shape is chosen to be the same as the vibration mode and only the results for the fundamental mode are presented even though the analysis is also applicable to the non-fundamental modes. It is found that the presence of imperfection amplitude of only half the plate thickness may significantly raise the linear vibration frequencies and change the inherent hard-spring character of composite plates to that of soft-spring behavior. The effects of fibre angles and number of layers for angle-ply rectangular plates and Young's modulus ratios and number of

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layers for graphite–epoxy, glass–epoxy and boron–epoxy cross-ply plates (which are likely to be encountered in practice [14, 23–26]) are examined. In passing, the effects of geometric imperfections on the linear frequencies of angle-ply rectangular plates with pre-load can be found in a separate paper [25].

The linear and nonlinear vibration behavior of the anti-symmetric angle-ply rectangular (square or non-square) plates are independent of the sign of the imperfection amplitude since from the physical point of view, such a sign change represents the same structure. The same remark on the sign independence of the imperfection amplitude can be made on vibrations of anti-symmetric cross-ply square plates. However, for non-square rectangular anti-symmetric cross-ply plates, the vibration problem may depend on the sign of the imperfection amplitude since a careful physical examination will show that such a sign change represents a different structure [26]. (Note that ref. [26] deals with buckling and postbuckling but not vibration.)

Of the numerous papers which deal with large-amplitude vibrations of laminated perfect plates (with no geometric imperfection) mentioned in Chia's book [14] and in review articles by Bert [15] and Leissa [16], All of them predicted hard-spring behavior. Some of them even employed highly accurate shell theory which includes transverse shear and rotary inertia, incorporating the effects of irregular boundary and variable thickness. This paper is the first in the open literature to point out that if these laminated structures were actually tested in the laboratory or in full-scale testing, it is possible that one may encounter soft-spring vibration behavior. This is because unavoidable geometric imperfections of the order of half the plate thickness are usually present due to manufacturing or other difficulties.

2. GOVERNING DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

The Von Kármán type nonlinear dynamic equilibrium and compatibility equations for a laminated thin plate written in terms of an out-of-plane displacement W and a stress function F valid for moderately large deflection are, respectively, (Stavsky and Hoff [27], Tennyson *et al.* [28, 29] and Chia [14]).

$$L_{D^*}(W)+L_{B^*}(F)+\rho W_{,\bar{t}\bar{t}}-Q(X,Y,\bar{t})$$
$$=F_{,YY}(W+W_0)_{,XX}+F_{,XX}(W+W_0)_{,YY}-2F_{,XY}(W+W_0)_{,XY} \tag{1}$$

$$L_{A^*}(F)-L_{B^*}(W)=(W_{,XY})^2+2W_{0,XY}W_{,XY}-(W+W_0)_{,XX}W_{,YY}-W_{0,YY}W_{,XX}. \tag{2}$$

In the above, ρ is the mass of the plate per unit area, $\bar{t}$ is time, X and Y are the in-plane coordinates, $Q(X, Y, \bar{t})$ is the forcing function, $W_0(X, Y)$ is the initial geometric imperfection and the linear differential operators $L_{A^*}(\ )$, $L_{B^*}(\ )$ and $L_{D^*}(\ )$ are defined in ref. [28] by setting the radius of the cylindrical shell to infinity. The strains and moments are related to the stresses and curvatures by

$$[\varepsilon_x, \varepsilon_y, \gamma_{xy}, M_x, M_y, M_{xy}]^T=\begin{bmatrix}[A^*_{ij}] & [B^*_{ij}]\\ [-B^*_{ij}] & [D^*_{ij}]\end{bmatrix}[N_x, N_y, N_{xy}, -W_{,XX}, -W_{,YY}, -2W_{,XY}]^T \tag{3}$$

$$[A^*_{ij}]=[A_{ij}]^{-1}, [B^*_{ij}]=-[A_{ij}]^{-1}[B_{ij}], [D^*_{ij}]=[D_{ij}]-[B_{ij}][A_{ij}]^{-1}[B_{ij}], \tag{4}$$

where $[A_{ij}]$, $[B_{ij}]$, $[D_{ij}]$, $[A^*_{ij}]$ and $[D^*_{ij}]$ are symmetric matrices and in general, $[B^*_{ij}]$ is not a symmetric matrix. Introducing the non-dimensional quantities a^*_{ij}, b^*_{ij}, d^*_{ij}, w, w_0, f, x, y, t and q defined by,

$$\begin{aligned}(a^*_{ij}, b^*_{ij}, d^*_{ij}) &= (EhA^*_{ij}, B^*_{ij}/h, D^*_{ij}/(Eh^3))\\ (w, w_0) &= (W/h, W_0/h), f=F/(Eh^3),\\ (x, y) &= (X/B, Y/B), t=\bar{t}\omega_r,\\ q(x,y,t) &= (Eh^4\pi^4/B^4)Q(X,Y,\bar{t}), (\omega_r)^2=Eh^3\pi^4/(\rho B^4),\end{aligned} \tag{5}$$

where h is the total thickness of the laminated plate, ω_r is the reference frequency, B is the width of the plate and E is a quantity which has unit force per square length (the choice for E usually depends on the unit used in the possibly given A^*_{ij}, B^*_{ij}, D^*_{ij} coefficients), one obtains,

$$L_{d^*}(w)+L_{b^*}(f)+\pi^4 w_{,tt}-\pi^4 q(x,y,t)$$
$$=f_{,yy}(w+w_0)_{,xx}+f_{,xx}(w+w_0)_{,yy}-2f_{,xy}(w+w_0)_{,xy} \tag{6}$$

$$L_{a^*}(f)-L_{b^*}(w)=(w+2w_0)_{,xy}w_{xy}-(w+w_0)_{,xx}w_{yy}-w_{0,yy}w_{,xx}. \tag{7}$$

For angle-ply plates, the non-dimensional linear operators are defined to be [25],

$$L_{a^*}(\ )=a^*_{22}(\ )_{,xxxx}+(2a^*_{12}+a^*_{66})(\ )_{,xxyy}+a^*_{11}(\ )_{,yyyy} \tag{8}$$

$$L_{d^*}(\ )=d^*_{11}(\ )_{,xxxx}+(2)(d^*_{12}+2d^*_{66})(\ )_{,xxyy}+d^*_{22}(\ )_{,yyyy} \tag{9}$$

$$L_{b^*}(\ )=(2b^*_{26}-b^*_{61})(\ )_{,xxxy}+(2b^*_{16}-b^*_{62})(\ )_{,xyyy}. \tag{10}$$

For cross-ply plates, the $L_{a^*}(\ )$ and $L_{d^*}(\ )$ operators are identical to that for angle-ply plates while the bending–stretching linear operator $L_{b^*}(\ )$ is replaced by $L_{\beta^*}(\ )$, where [26],

$$L_{\beta^*}(\ ) = b^*_{21}(\ )_{,xxxx} + (b^*_{11} + b^*_{22} - 2b^*_{66})(\ )_{,xxyy} + b^*_{12}(\ )_{,yyyy}. \tag{11}$$

For anti-symmetric angle-ply plates, one obtains

$$A_{16} = A_{26} = B_{11} = B_{12} = B_{22} = B_{66} = D_{16} = D_{26} = 0 \tag{12}$$

while for anti-symmetric cross-ply plates,

$$A_{16} = A_{26} = B_{16} = B_{26} = D_{16} = D_{26} = 0. \tag{13}$$

3. SIMPLY SUPPORTED IMPERFECT ANGLE-PLY PLATES

In a one-mode approximate analysis, the vibration mode which satisfies the simply supported boundary conditions ($w = 0$, $M_x = 0$ at $X = 0, L$ and $w = 0$, $M_y = 0$ at $Y = 0, B$) along all four edges and the geometric imperfection and forcing function are,

$$[w(x, y, t), w_0(x, y), q(x, y, t)] = [w(t), \mu, q(t)] \sin(M\pi x) \sin(n\pi y), \tag{14}$$

where μ is the imperfection amplitude normalised, with respect to the total thickness of the laminated plate, $M = mB/L$, L is the length of the plate and m and n are the number of half-waves in the x and y directions, respectively. Substituting the vibration mode and the imperfection into the nonlinear compatibility equation, the stress function which satisfies this nonlinear differential equation exactly is,

$$f(x, y, t) = [w(t)^2 + 2\mu w(t)][c_0 w(t) \cos(M\pi x) \cos(n\pi y) + c_1 \cos(2M\pi x) + c_2 \cos(2n\pi y) + (e_1 x^2/2) + (e_2 y^2/2)], \tag{15}$$

where

$$\begin{aligned} c_0 &= -C_{b^*}(M, n)/C_{a^*}(M, n) \\ c_1 &= n^2/(32M^2 a^*_{22}) \\ c_2 &= M^2/(32n^2 a^*_{11}) \end{aligned} \tag{16}$$

and the functions $C_{a^*}(P, Q)$ and $C_{b^*}(P, Q)$ are defined to be

$$\begin{aligned} C_{a^*}(P, Q) &= (2b^*_{26} - b^*_{61})(-P^3 Q) + (2b^*_{16} - b^*_{62})(PQ^3) \\ C_{b^*}(P, Q) &= a^*_{22} P^4 + (2a^*_{12} + a^*_{66}) P^2 Q^2 + a^*_{11} Q^4. \end{aligned} \tag{17}$$

Furthermore, it can be seen by inspection that there is no in-plane shear along all four edges ($f_{,xy} = 0$). If each of the four edges is permitted to move in the direction perpendicular to the edge (that is, in-plane movable), one obtains,

$$e_1 = e_2 = 0. \tag{18}$$

Since the mixed formulation is used, the in-plane displacement boundary conditions can only be satisfied on the average. Before one considers the in-plane immovable boundary conditions, it is desirable to write down the stress function-displacements relations in the form (U and V are the in-plane displacements in the x and y directions, respectively).

$$\begin{aligned} f_{,yy} &= a_{11}[u_{,x} + \tfrac{1}{2}(w_{,x}^2 + 2w_{0,x} w_{,x})] + a_{12}[v_{,y} + \tfrac{1}{2}(w_{,y}^2 + 2w_{0,y} w_{,y})] - 2b_{16} w_{,xy} \\ f_{,xx} &= a_{21}[u_{,x} + \tfrac{1}{2}(w_{,x}^2 + 2w_{0,x} w_{,x})] + a_{22}[v_{,y} + \tfrac{1}{2}(w_{,y}^2 + 2w_{0,y} w_{,y})] - 2b_{26} w_{,xy}, \end{aligned} \tag{19}$$

where (a_{ij} and b_{ij} are not to be confused with a^*_{ij} and b^*_{ij})

$$(u, v) = (B/h^2)(U, V), \ (a_{ij}, b_{ij}) = [A_{ij}/(Eh), B_{ij}/(Eh^2)]. \tag{20}$$

Thus, for a rectangular plate with four in-plane immovable edges, the quantities $u_{,x}$ and $v_{,y}$ should not contain constant terms (that is, terms which are independent of the x and y coordinates) so that one obtains,

$$\begin{aligned} e_1 &= (a_{21} M^2 + a_{22} n^2)(\pi^2/8) \\ e_2 &= (a_{11} M^2 + a_{12} n^2)(\pi^2/8). \end{aligned} \tag{21}$$

Substituting $w(x, y, t)$, $w_0(x, y)$ and $f(x, y, t)$ into the nonlinear dynamic equilibrium equation for angle-ply plates, and performing the Gelerkin procedure (that is, multiplying both sides by $\sin(M\pi x)\sin(n\pi y)$ and then integrating over the plate area), one obtains,

$$\begin{aligned} &w(t)_{,tt} + \{C_{d^*}(M, n) + [C_{b^*}(M, n)^2/C_{a^*}(M, n)]\} w(t) - q(t) \\ &\quad = [w(t)^3 + 3\mu w(t)^2 + 2\mu^2 w(t)](-e_0), \end{aligned} \tag{22}$$

where

$$\begin{aligned} C_{d^*}(P, Q) &= d^*_{11} P^4 + 2(d^*_{12} + 2d^*_{66}) P^2 Q^2 + d^*_{22} Q^4 \\ e_0 &= [(M^2 e_2 + n^2 e_1)/\pi^2] + (2M^2 n^2)(c_1 + c_2). \end{aligned} \tag{23}$$

Thus, the above Duffing-type nonlinear ordinary differential equation in time can be written in standard form (Hui [18–20])

$$w(t)_{,tt} + k\,w(t) + (\varepsilon k a_2) w(t)^2 + (\varepsilon k) w(t)^3 = q(t), \tag{24}$$

where

$$k = C_{d^*}(M, n) + [C_{b^*}(M, n)^2/C_{a^*}(M, n)] + 2\mu^2 e_0$$
$$\varepsilon k a_2 = 3\mu e_0, \varepsilon k = e_0. \tag{25}$$

It should be noted that the linear vibration frequency $k^{\frac{1}{2}}$ is independent of the sign of the imperfection amplitude. In the special case of a perfect angle-ply plate ($\mu = 0$), the in-plane boundary conditions (which enters the problem via the constant e_0) have no effect on the linear vibration frequency.

Using Linstedt's perturbation technique [19, 30], the nonlinear frequency is related to the vibration amplitude in the case of free vibration ($q(t) = 0$).

$$\omega(\text{nonlinear})/\omega(\text{linear}) = 1 + rA^2 + \dots, \tag{26a}$$

where

$$r = (3\varepsilon/8) - (5a_2^2\varepsilon^2/12) = (3\varepsilon/8)[1 - (10a_2^2\varepsilon/9)]. \tag{26b}$$

Thus, at least for sufficiently small vibration amplitude, the nonlinear vibration problem is classifed as 'hard spring' for $r > 0$ and 'soft spring' for $r < 0$. A larger value of the magnitude of r indicates (i) a more pronounced hard-spring behavior if r is positive or (ii) a more pronounced soft-spring behavior if r is negative.

4. CLAMPED IMPERFECT ANGLE-PLY PLATES

As the objective of the present paper is to investigate the influence of geometric imperfections and in-plane boundary conditions on the linear and nonlinear vibration behavior in a relatively straightforward manner, it is felt that the major trends can be obtained from a one-mode approximate analysis. The vibration mode, the geometric imperfection and the forcing function which satisfy the clamped boundary conditions are [14],

$$[w(x, y, t), w_0(x, y), q(x, y, t)] = \tfrac{1}{4}[w(t), \mu, q(t)][1 - \cos(2M\pi x)][1 - \cos(2n\pi y)], \tag{27}$$

where μ is the imperfection amplitude normalised, with respect to the total laminated plate thickness. The stress function which satisfies the nonlinear compatibility equation exactly for a clamped angle-ply plate is,

$$\begin{aligned} f(x, y, t) = {} & k_0 w(t) \sin(2M\pi x) \sin(2n\pi y) \\ & + [w(t)^2 + 2\mu w(t)]\{k_1 \cos(2M\pi x) + k_2 \cos(2n\pi y) + k_3 \cos(4M\pi x) + k_4 \cos(4n\pi y) \\ & + k_5 \cos(2M\pi x)\cos(2n\pi y) + k_6 \cos(2M\pi x)\cos(4n\pi y) \\ & + k_7 \cos(4M\pi x)\cos(2n\pi y) + (e_3 x^2/2) + (e_4 y^2/2)\}, \end{aligned} \tag{28a}$$

where

$$\begin{aligned} k_0 &= C_{b^*}(2M, 2n)/[-4C_{a^*}(2M, 2n)] \\ k_1 &= n^2/(32M^2 a_{22}^*), k_2 = M^2/(32n^2 a_{11}^*)] \\ k_3 &= -n^2/[(16)(32M^2 a_{22}^*)], k_4 = -M^2/[(16)(32n^2 a_{11}^*)] \\ k_5 &= -M^2 n^2/C_{a^*}(2M, 2n), k_6 = M^2 n^2/[2C_{a^*}(2M, 4n)] \\ k_7 &= M^2 n^2/[2C_{a^*}(4M, 2n)]. \end{aligned} \tag{28b}$$

Substituting $w(x, y, t)$, $w_0(x, y)$ and $f(x, y, t)$ into the nonlinear dynamic equilibrium equation for angle-ply plates and then performing the Galerkin procedure (multiplying both sides by $[1 - \cos(2m\pi x)] \cdot [1 - \cos(2n\pi y)]$ and integrating over the plate area), one obtains a nonlinear ordinary differential equation in time which can be written in standard form (see equation 24) where,

$$k = 2\mu^2 R + (4/9)[-k_0 C_{b^*}(2M, 2n) + (1/4)C_{d^*}(2M, 2n) + 8M^4 d_{11}^* + 8n^4 d_{22}^*]$$
$$\varepsilon k a_2 = 3\mu R, \varepsilon k = R. \tag{29}$$

In the above, the quantity R is defined to be,

$$R = [16/9\pi^4)][4M^2 n^2 \pi^4 k_8 + (3M^2\pi^2 e_4/4) + (3n^2\pi^2 e_3/4)], \tag{30}$$

where

$$k_8 = \tfrac{1}{2}\{[k_2 - k_4 - k_5 + k_6 + (k_7/2)] + [k_1 - k_3 - k_5 + (k_6/2) + (k_7/2)] + (k_5 - k_6 - k_7)\}.$$

Furthermore, the constants e_3 and e_4 depend on the in-plane boundary conditions. For all four edges in-plane immovable, one obtains,

$$e_3 = (3\pi^2/32)(a_{21}M^2 + a_{22}n^2)$$
$$e_4 = (3\pi^2/32)(a_{11}M^2 + a_{12}n^2), \tag{31}$$

while $e_3 = e_4 = 0$ for all four edges in-plane movable.

5. CLAMPED IMPERFECT CROSS-PLY PLATES

In a similar manner, the vibration mode, the geometric imperfection and the forcing function for clamped cross-ply plates are chosen to be [14],

$$[w(x, y, t), w_0(x, y), q(x, y, t)] = [w(t), \mu, q(t)]\tfrac{1}{4}[1 - \cos(2M\pi x)][1 - \cos(2n\pi y)]. \tag{32}$$

The stress function which satisfies the nonlinear compatibility equation exactly is,

$$\begin{aligned} f(x, y, t) = {} & r_0 w(t)\cos(2M\pi x)\cos(2n\pi y) + r_8 w(t)\cos(2M\pi x) + r_9 w(t)\cos(2n\pi y) \\ & + [w(t)^2 + 2\mu w(t)]\{r_1\cos(2M\pi x) + r_2\cos(2n\pi y) + r_3\cos(4M\pi x) + r_4\cos(4n\pi y) \\ & + r_5\cos(2M\pi x)\cos(2n\pi y) + r_6\cos(2M\pi x)\cos(4n\pi y) \\ & + r_7\cos(4M\pi x)\cos(2n\pi y) + (e_5 x^2/2) + (e_6 y^2/2)\}, \end{aligned} \tag{33}$$

where

$$\begin{aligned} r_0 &= C_{\beta^*}(2M, 2n)/[4C_{a^*}(2M, 2n)] \\ r_8 &= -b^*_{21}/(4a^*_{22}),\ r_9 = -b^*_{12}/(4a^*_{11}) \\ C_{\beta^*}(P, Q) &= b^*_{21}P^4 + (b^*_{11} + b^*_{22} - 2b^*_{66})P^2Q^2 + b^*_{12}Q^4 \end{aligned} \tag{34}$$

and $r_1 = k_1, r_2 = k_2, \ldots, r_7 = k_7$ provided that a^*_{ij} coefficients are those appropriate for cross-ply plates.

Substituting $w(x, y, t)$, $w_0(x, y)$ and $q(x, y, t)$ into the nonlinear dynamic equilibrium equation for cross-ply plates and carrying out the Galerkin procedure, with respect to $[1-\cos(2M\pi x)]\cdot[1-\cos(2n\pi y)]$, one obtains a nonlinear ordinary differential equation in time in the form,

$$\begin{aligned} & w(t)_{,tt} + \gamma_1 w(t) + [w(t)^2 + 2\mu w(t)]\beta_1 + [w(t)^2 + \mu w(t)]r_{10} \\ & \quad + [w(t)^3 + 3\mu w(t)^2 + 2\mu^2 w(t)]R = q(t), \end{aligned} \tag{35}$$

where (the constants γ_1, β_1, r_{10} and R are independent of the imperfection amplitude μ),

$$\begin{aligned} \gamma_1 &= (16/9)[-8M^4 b^*_{21} r_8 - 8n^4 b^*_{12} r_9 + (r_0/4)C_{\beta^*}(2M, 2n) + (1/16)C_{d^*}(2M, 2n) \\ &\quad + 2d^*_{11}M^4 + 2d^*_{22}n^4] \\ \beta_1 &= (16/9)[-8M^4 b^*_{21} r_1 - 8n^4 b^*_{12} r_2 + (r_5/4)C_{\beta^*}(2M, 2n)] \\ r_{10} &= (16M^2n^2/9)(2)(r_0 - r_9 - r_8) \\ R &= -4M^2n^2\pi^4 k_8 - (3M^2\pi^2 e_6/4) - (3n^2\pi^2 e_5/4). \end{aligned} \tag{36}$$

In the above, the constant k_8 is defined in equation (30). Further, for all four edges in-plane immovable,

$$e_5 = e_3,\ e_6 = e_4 \tag{37}$$

and $e_5 = e_6 = 0$ for all four edges in-plane movable. Re-arranging terms, the above differential equation can be written in the standard Duffing-type equation (equation 24) where,

$$\begin{aligned} k &= \gamma_1 + 2\mu\beta_1 + \mu r_{10} + 2\mu^2 R \\ \varepsilon k a_2 &= \beta_1 + r_{10} + 3\mu R,\ \varepsilon k = R. \end{aligned} \tag{38}$$

6. DISCUSSION OF RESULTS

The material parameters for the anti-symmetric angle-ply laminated plates are chosen to be those appropriate for graphite-epoxy composites with [23–24],

$$E_1/E_2 = 40,\quad G_{12}/E_2 = 0.5,\quad \nu_{12} = 0.25,\quad \nu_{21} = \nu_{12}(E_2/E_1). \tag{39}$$

The numerical results for all the curves are presented only for the vibration wave number $M = mB/L = 1$ and $n = 1$ and it should be noted that the effects of aspect ratio L/B and m can be studied via a single parameter M. The frequencies presented are normalised with respect to $\omega(\theta = \mu = 0$, s.s.); that is, the frequency of simply supported, perfect angle-ply plates with zero fibre angle. It should be noted that,

$$\begin{aligned} \Omega(\text{Chia}) &= \omega B^2[\rho/(E_2h^3)]^{\frac{1}{2}} = \pi^2 k^{\frac{1}{2}} \\ \Omega(\text{Chia, at } \theta = \mu = 0, \text{ s.s.}) &= 1.9053\pi^2 = 18.805. \end{aligned} \tag{40}$$

Figure 1(a) shows a graph of normalised frequency $\omega/\omega(\theta = \mu = 0$, s.s.) versus imperfection amplitude μ for simply supported anti-symmetric angle-ply plates with all edges in-plane movable ($e_1 = e_2 = 0$). The fibre angles under consideration are 0°, 15°, 30°, 45° and the effects of bending–stretching coupling can be seen by comparing the orthotropic plates (infinite layers) with the two-layer plates. Clearly, the presence of geometric imperfections of the order of the total thickness of the laminated plate may significantly raise the linear vibration frequencies. Further, the increases in frequencies are much more pronounced for the 0° and 15° plates than for the 30° and 45° plates. The corresponding non-linearity indicator r versus the imperfection amplitude μ curves are plotted in Fig. 1(b). For sufficiently small values of the imperfection amplitude, the angle-ply plates display hard-spring (r positive) behavior while for larger value of μ, the plates exhibit soft-spring (r negative) behavior. The magnitudes of the nonlinearity indicator (larger magnitude means more pronounced hard-spring or soft-spring behavior) are much larger for the 0° and 15° plates than for the 30° and 45° plates.

Similarly, the normalised frequency versus imperfection amplitude curve for simply supported angle-ply plates with all edges in-plane immovable are presented in Fig. 2(a). It is important to note that the in-plane boundary conditions have no effect on the linear frequencies. Here, the increases in frequencies are more pronounced than those for the plates with all edges movable. The corresponding curves of nonlinearity indicator r versus imperfection amplitude are shown in Fig. 2(b). It can be seen that the magnitude of r (note the change in scale in the graph) is much larger than those for the in-plane movable edges. Finally, the transitional value of the imperfection amplitude at which r changes sign μ_t occurs at $\mu_t < 0.4$. These values are much smaller than those for the in-plane movable plates.

Figure 3(a) shows a graph of the normalised frequency $\omega/\omega(\theta = \mu = 0$, s.s.) versus the imperfection amplitude for clamped angle-ply plates with all edges in-plane movable ($e_3 = e_4 = 0$). Again, the 0° and 15° plate frequencies increase with imperfection amplitude much faster than the 30° and 45° plates. Of interest is that for the orthotropic plates (infinite layers), all the curves pass through $\Omega(\text{Chia}) = 4.3152\pi^2 = 42.589$. The corresponding nonlinearity

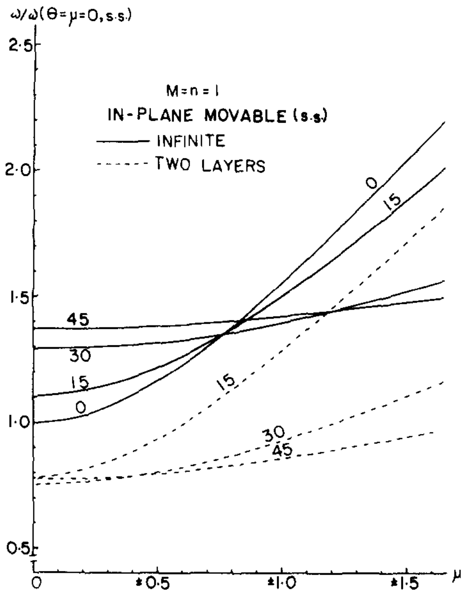

FIG. 1(a). Normalised linear frequency vs imperfection amplitude for simply supported angle-ply rectangular plates with all edges in-plane movable ($e_1 = e_2 = 0$).

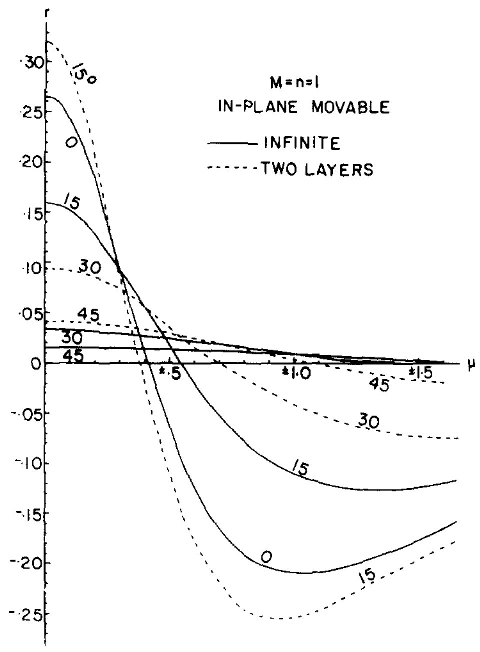

FIG. 1(b). Nonlinearity indicator vs imperfection amplitude for simply angle-ply rectangular plates with all edges in-plane movable ($e_1 = e_2 = 0$).

indicator r versus imperfection amplitude curves are depicted in Fig. 3(b). In general, the magnitudes of r for the 0° and 15° plates are larger than those for the 30° and 45° plates. Figure 4(a) shows a graph of the normalised frequency versus imperfection amplitude for clamped angle-ply plates with all edges in-plane immovable. In general, the frequencies of the two-layer plates increase with imperfection amplitude faster than those for the orthotropic plates. Finally, the corresponding nonlinearity indicator versus imperfection amplitude curves are presented in Fig. 4(b).

Three types of materials are selected for cross-ply laminated plates [15]. They are graphite–epoxy ($E_1/E_2 = 40.0$, $G_{12}/E_2 = 0.5$ and $\nu_{12} = 0.25$), glass–epoxy ($E_1/E_2 = 3.0$, $G_{12}/E_2 = 0.5$ and $\nu_{12} = 0.25$) and boron–epoxy ($E_1/E_2 = 10.0$, $G_{12}/E_2 = 1/3$ and $\nu_{12} = 0.22$). Figure 5(a) shows a graph of the linear vibration frequency $k^{\frac{1}{2}} = \Omega$ (Chia)$/\pi^2 = \omega/\omega_r$ versus positive values of the imperfection amplitude for clamped graphite–epoxy cross-ply plates ($M = n = 1$). A very small increase in the vibration frequency for two-layer plates (but not infinite-layer) is detected for the corresponding negative values of the imperfection amplitude (too small to be plotted). Thus, the linear

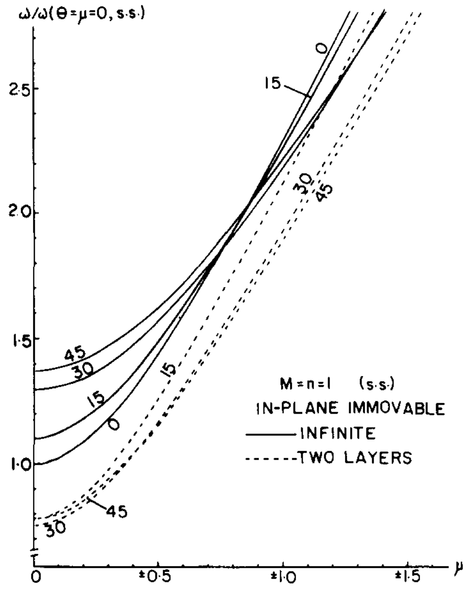

FIG. 2(a). Normalised linear frequency vs imperfection amplitude for simply angle-ply rectangular plates with all edges in-plane immovable.

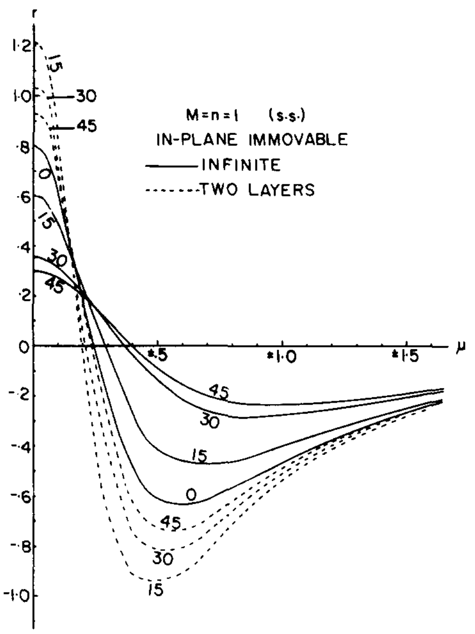

FIG. 2(b). Nonlinearity indicator vs imperfection amplitude for simply angle-ply rectangular plates with all edges in-plane immovable.

frequency (and the nonlinearity indicator which will be shown later) has a slight dependence on the sign of the imperfection amplitude. Again, the frequencies of the plates with all edges in-plane immovable increase more with imperfection amplitude than those for the plates with all edges in-plane movable. The corresponding curves of nonlinearity indicator versus imperfection amplitude are shown in Fig. 5(b) for clamped graphite–epoxy cross-ply plates. It should be cautioned that for non-square ($M \neq 1$) rectangular cross-ply plates (not treated here), the problem may depend on the sign of the imperfection amplitude since a sign change corresponds to a different structure.

Finally, the linear frequency $k^{\frac{1}{2}}$ versus positive values of the imperfection amplitude curves for clamped glass–epoxy and clamped boron–epoxy cross-ply plates are presented in Fig. 6(a) with $M = n = 1$. The increases in linear frequencies with imperfection amplitude for these plates are less pronounced than those for graphite–epoxy cross-ply plates. The corresponding nonlinearity indicator r versus positive values of imperfection amplitude curves for clamped glass–epoxy and clamped boron–epoxy cross-ply plates are shown in Fig. 6(b). In general, the

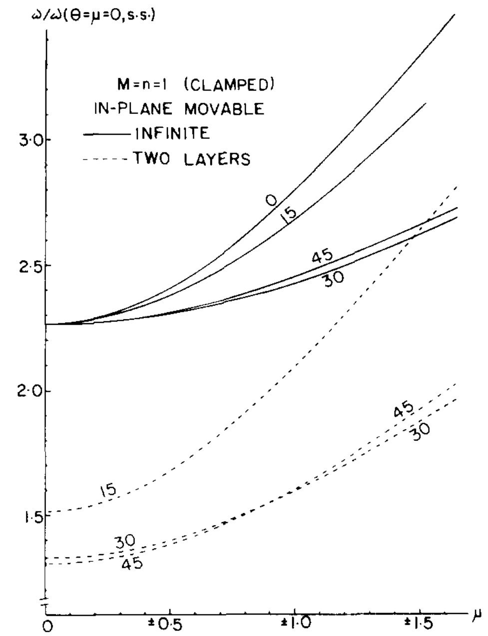

FIG. 3(a). Normalised linear frequency vs imperfection amplitude for clamped angle-ply rectangular plates with all edges in-plane movable ($e_3 = e_4 = 0$).

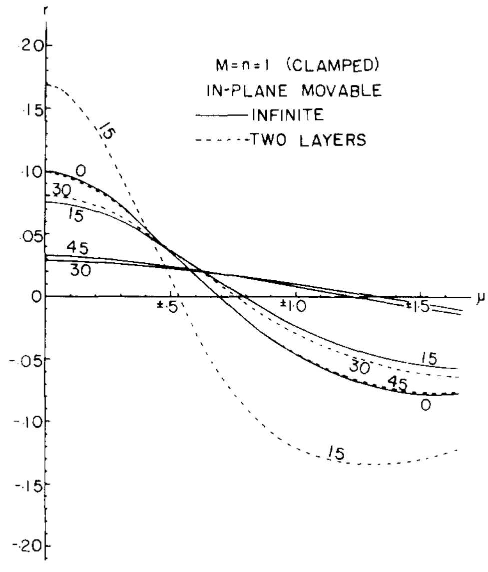

FIG. 3(b). Nonlinearity indicator vs imperfection amplitude for clamped angle-ply rectangular plates with all edges in-plane movable ($e_3 = e_4 = 0$).

magnitudes of r are smaller than those for the graphite–epoxy cross-ply plates. Again, very small increases in frequency and slightly more negative values of r are found for the corresponding negative values of the imperfection amplitude.

As a check on the analysis, the linear frquencies for angle- and cross-ply plates in the present one-mode approximate analysis are found to be slightly (approx. 1%) higher than those obtained from a four-mode analysis [14]. These small discrepancies (on the high side) are expected since the nonlinear compatibility equation is satisfied exactly in the present analysis, thus, the computed frequencies are upper-bound frequencies.

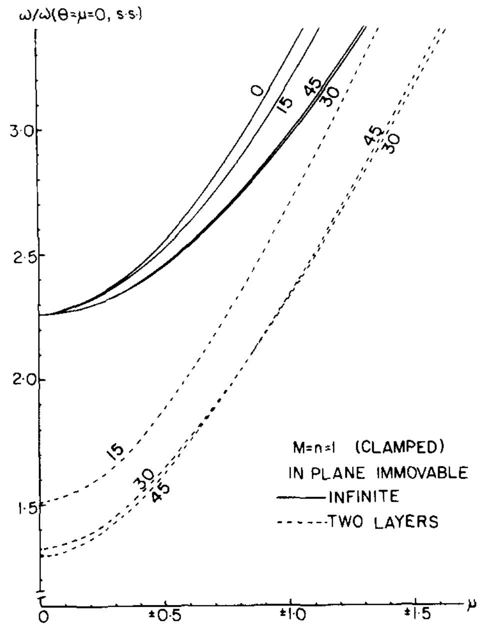

FIG. 4(a). Normalised linear frequency vs imperfection amplitude for clamped angle-ply rectangular plates with all edges in-plane immovable.

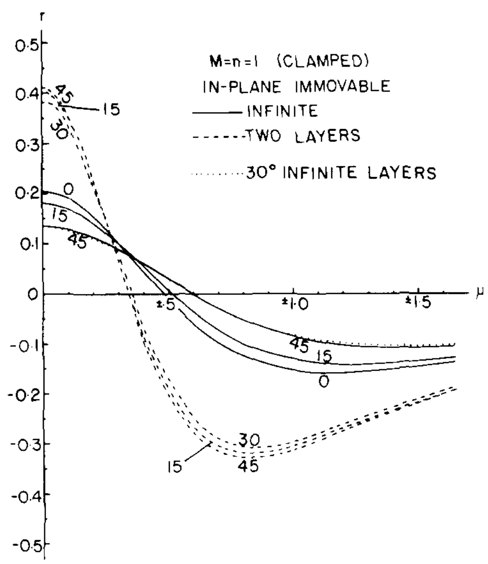

FIG. 4(b). Nonlinearity indicator vs imperfection amplitude for clamped angle-ply rectangular plates with all edges in-plane immovable.

7. CONCLUDING REMARKS

The effects of geometric imperfections and in-plane boundary conditions on the linear and large amplitude vibration behavior of laminated rectangular plates have been examined. The equal-thickness antisymmetrically laminated plates being considered are graphite–epoxy angle-ply plates (all edges simply supported or clamped) and graphite–epoxy, glass–epoxy and boron–epoxy cross-ply plates (all edges clamped). It was found that the presence of unavoidable geometric imperfections of the order of a fraction of the total laminated plate

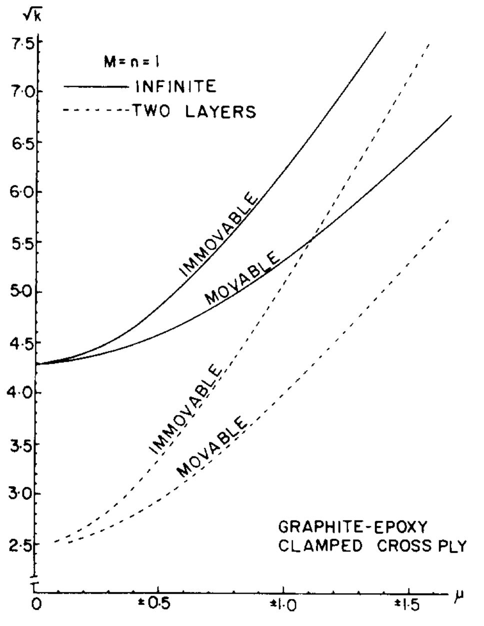

FIG. 5(a). Linear frequency $k^{\frac{1}{2}}$ vs imperfection amplitude for clamped graphite–epoxy cross-ply rectangular plates.

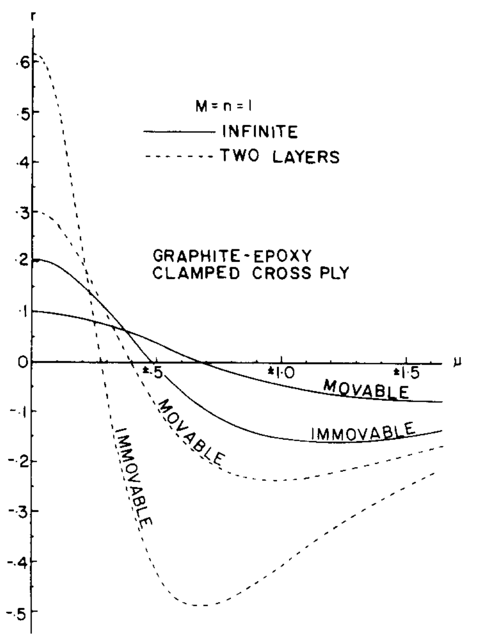

FIG. 5(b). Nonlinearity indicator vs imperfection amplitude for clamped graphite–epoxy cross-ply rectangular plates.

thickness may significantly raise the linear vibration frequencies. Further, they may change the inherent hard-spring non-linear vibration character of these plates to soft-spring behavior. It appears that this paper is the first in the open literature to deal with the effects of geometric imperfections on large amplitude vibrations of laminated plates, incorporating the possibility of bending–stretching couplings. Since the influence of geometric imperfections

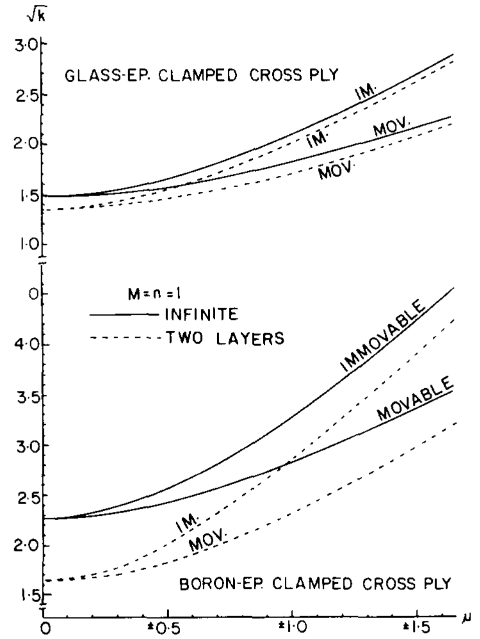

FIG. 6(a). Linear frequency $k^{\frac{1}{2}}$ vs imperfection amplitude for clamped glass–epoxy and boron–epoxy cross-ply rectangular plates.

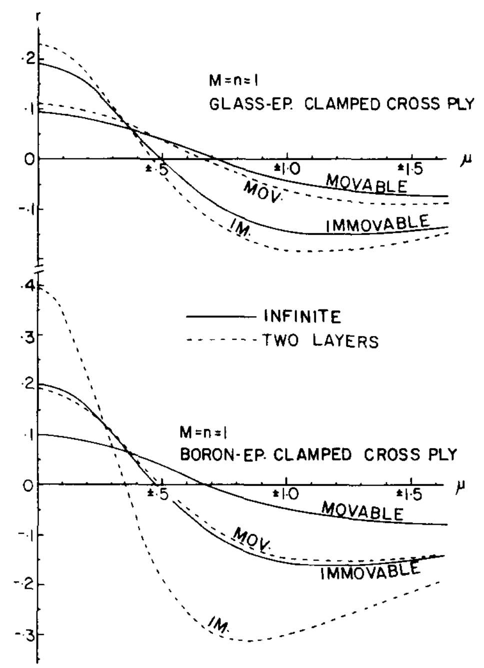

FIG. 6(b). Nonlinearity indicator vs imperfection amplitude for clamped glass–epoxy and boron–epoxy cross-ply rectangular plates.

on both the linear and nonlinear vibrations of isotropic homogeneous structures is found to be highly significant, one would tend to be reluctant to use composite structures since it is not clear that these imperfections will produce the same trends and if so, to what extent. The present results show that the effects of imperfections on vibrations of laminated structures generally produce the same trends but to a smaller extent for a wide variety of fiber orientations, stacking sequence and in-plane and out-of-plane boundary conditions.

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