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DESIGN OF BENEFICIAL GEOMETRIC IMPERFECTIONS FOR ELASTIC COLLAPSE OF THIN-WALLED BOX COLUMNS

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Abstract—This paper deals with the design of beneficial geometric imperfections for elastic collapse of thin-walled box columns of square cross-section under axial compression. From the point of view of high elastic post-collapse stiffness, it is desirable to force the column to collapse with a larger number of axial half-waves than with the preferred wave number which is approximately equal to the aspect ratio. By introducing a beneficial geometric imperfection with such a larger wave number and if the magnitude of such imperfection exceeds the transitional value, it is found that the column will collapse in the beneficial mode in the initial postbuckling (assume to be elastic) finite-deflection regime. Equilibrium paths of typical box columns are plotted and analyzed. The two-mode potential energy is found to fall into the category of a double-cusp catastrophe.

1. INTRODUCTION

Collapse of thin-walled columns is a topic that has received considerable attention over the past decades due to the common use of such columns against axial compression in many aerospace and mechanical structural applications. The finite deflection crushing of these columns has found an increasing number of applications in impact engineering problems such as aircraft, automobile and ship collisions. Unlike the numerous column buckling studies reported in the open literature, the aim of this paper is to design the thin-walled column with high elastic post-collapse stiffness rather than simply high buckling load.

Of particular interest is the elastic collapse of thin-walled box columns with square cross-section under axial compression (see Fig. 1). With respect to energy absorption of these columns, it is advantageous to design a mechanism to allow the column to collapse in as many waves as feasible in the direction of the axial load. One mechanism is to introduce a 'beneficial' geometric imperfection of specified shape and magnitude so that the structure will follow the equilibrium paths of this beneficial mode in the postbuckling as well as in the very large deflection regimes. Although there are many investigations on the buckling of box columns, the effects of geometric imperfections on the mode of collapse (which affects the energy absorption) has received minimal attention. It appears that the introduction of 'beneficial' geometric imperfection for collapse of thin-walled box columns has not been considered in the open literature.

However, the beneficial geometric imperfection will be beneficial only during the formation of the first fold up to the occurrence of the first touching. The formation of the subsequent folds is governed to the same extent by the buckling and postbuckling behavior of thin-walled structures as by the process of progressive plastic crushing. The length of the folding wave is generally smaller than that of the elastic buckling wavelength. Thus, the mode of progressive plastic collapse may not necessarily follow the imposed shape of initial imperfection profile. Despite the above limitations on the usefulness in the 'plastic' range, these imperfections are beneficial in the initial post-collapse 'elastic' range in a postbuckling or impact problem.

A classical comprehensive study on the coupled flexural–torsional elastic buckling of thin-walled columns was presented by Timoshenko [1]. A theoretical and experimental investigation of the local collapse of thin-walled square box columns was performed by Graves Smith and Sridharan [2]. Theoretical studies on the crushing of thin-walled box columns and the behavior of plate intersections were reported by Wierzbicki and Akerstrom [3] and Wierzbicki [4, 5] and comparisons were made with experiments performed by Magee and Thornton [6], among others. An excellent theoretical and experimental study on the dynamic axial crushing of square tubes was reported by Abramowicz and Jones [7].

Many other experimental investigations on the crushing of square tubes were reported in recent years. Meng *et al.* [8] examined the mean axial crushing load of square tubes using an incremental plasticity analysis which allowed for travelling plastic hinges and large deflection. An experimental study of interaction buckling strength of welded built-up box columns was carried out by Usami and Fukumoto [9]. Experimental data on the collapse characteristics and energy absorption of box columns were reported by Rawlings and Shapland [10], Mahmood and Paluszny [11], Murray [12] and Murase and Katori [13]. The above experimental results (particularly Meng *et al.* [8]) suggest that, from the energy absorption point of view, it is efficient to design a mechanism which will allow the column to collapse in many axial waves.

The present paper is concerned with the effects of beneficial geometric imperfections on the postbuckling strength of thin-walled square box columns. The analysis is based on a solution of the von Kármán nonlinear equilibrium and compatibility equations within the context of Koiter's multi-mode postbuckling theory (Koiter [14] and Hui [15]). The deflection modes and the second order fields are computed in closed form and the postbuckling coefficients of the two-mode problem are found from energy principles. The equilibrium paths of typical thin-walled box columns are plotted and analyzed. The results indicate that it is possible to introduce a beneficial geometric imperfection of magnitude greater than the transitional value so that the equilibrium paths will follow the beneficial mode collapse in the postbuckling as well as in the very large deflection regimes, leading to a more desirable energy absorption design of these columns. The two-mode potential energy is found to fall into the category of a double-cusp model.

2. GOVERNING DIFFERENTIAL EQUATIONS AND BUCKLING ANALYSIS

The present analysis is based on a solution of the von Kármán non-linear equilibrium and compatibility equations for plates written in terms of an out-of-plane displacement W and a stress function F valid for moderately large deflections (Hui and Leissa [16]),

$$(D)(W_{,XXXX}+W_{,YYYY}+2W_{,XXYY}) = F_{,YY}W_{,XX}+F_{,XX}W_{,YY}-2F_{,XY}W_{,XY} \quad (1)$$

$$[1/(Eh)](F_{,XXXX}+F_{,YYYY}+2F_{,XXYY}) = (W_{,XY})^2 - W_{,XX}W_{,YY}, \quad (2)$$

where E is Young's modulus, D is the flexural regidity, h is the plate thickness and X and Y are the in-plane coordinates (see Fig. 1). Introducing non-dimensional quantities,

$$(x, y) = (X, Y)(1/B), \quad w = W/h, \quad f = 2cF/(Eh^3) \quad (3)$$

where B is the width of the plate ($c = [3(1-\nu^2)]^{1/2}$ and ν is Poisson's ratio), the above nonlinear differential equations can be written in non-dimensional form,

$$w_{,xxxx}+w_{,yyyy}+2w_{,xxyy} = (2c)[f_{,yy}w_{,xx}+f_{,xx}w_{,yy}-2f_{,xy}w_{,xy}] \quad (4)$$

$$f_{,xxxx}+f_{,yyyy}+2f_{,xxyy} = (2c)[(w_{,xy})^2-w_{,xx}w_{,yy}]. \quad (5)$$

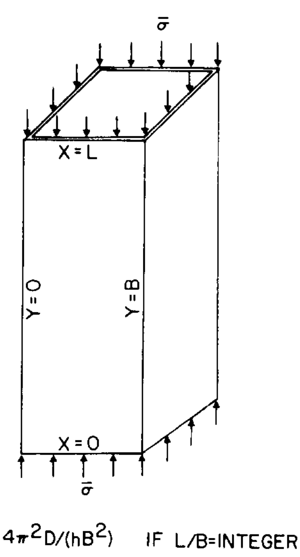

FIG. 1. Thin-walled square box column under axial compression.

For a rectangular plate under axial compression, the total deflection and stress function can be written in the form (Hui and Hansen [17] and Hui [15]),

$$w = 0+(\xi_1 w_1+\xi_2 w_2)+0 = 0+w_c+0$$
$$f = f_p+0+(\xi_1^2 f_{11}+\xi_2^2 f_{22}+2\xi_1\xi_2 f_{12}) = f_p+0+f_{11}, \tag{6}$$

where ξ_1 and ξ_2 are the amplitudes of the buckling mode and the beneficial mode respectively and f_p is the prebuckling stress function. Further, it is found that,

$$w_p = 0 \quad f_1 = f_2 = 0 \quad \text{and } w_{11} = w_{22} = w_{12} = 0. \tag{7}$$

Thus, the linearized differential equation is,

$$w_{c,\,xxxx} + w_{c,\,yyyy} + 2w_{c,\,xxyy} + (\pi^2\sigma)w_{c,\,xx} = 0, \tag{8}$$

where ($\bar{\sigma}$ is the axial stress),

$$\sigma = k(\text{Timoshenko}) = \bar{\sigma}hB^2/(\pi^2 D) = (-2c/\pi^2)f_{p,\,yy}. \tag{9}$$

Assuming that the rectangular plate is simply supported along all four edges, the two modes can be written in the form,

$$w_1(x,\,y) = \xi_1 \sin(M_1\pi x)\sin(\pi y)$$
$$w_2(x,\,y) = \xi_2 \sin(M_2\pi x)\sin(\pi y), \tag{10}$$

where $M_1 = \; = m_1 B/L$ and $M_2 = m_2 B/L$ and the positive integers m_1 and m_2 are the number of axial half waves for the first and second modes, respectively. The corresponding critical loads are,

$$\sigma_1 = (M_1^2+1)^2/(M_1^2)$$
$$\sigma_2 = (M_2^2+1)^2/(M_2^2). \tag{11}$$

In the special case of an infinitely long rectangular plate or if the aspect ratio L/B is a positive integer, the buckling load and the corresponding wave number are, respectively,

$$\sigma_c = 4$$
$$M_c = 1, \tag{12}$$

which agrees with the values reported by Timoshenko and Gere [18].

3. SECOND ORDER FIELDS

According to Koiter's multi-mode postbuckling theory, it is necessary to compute the second order fields in addition to the buckling modes w_1 and w_2 (note that in the present analysis, these two competing modes need not be simultaneous). The governing differential equation for the second order field is,

$$f_{11,\,xxxx}+f_{11,\,yyyy}+2f_{11,\,xxyy} = (2c)[(w_{c,\,xy})^2-(w_{c,\,xx})(w_{c,\,yy})] \tag{13}$$

so that the differential equation for $f_{11}(x,\,y)$ is,

$$f_{11,\,xxxx}+f_{11,\,yyyy}+2f_{11,\,xxyy} = (2c)[(w_{1,\,xy})^2-(w_{1,\,xx})(w_{1,\,yy})]. \tag{14}$$

Based on the right-hand side of equation (14) the solution for $f_{11}(x,\,y)$ is

$$f_{11}(x,\,y) = e_1\cos(2M_1\pi x)+e_2\cos(2\pi y), \tag{15}$$

where

$$e_1 = c/(16M_1^2)$$
$$e_2 = cM_1^2/16. \tag{16}$$

The in-plane boundary conditions are such that there is no in-plane shear stress along any of the four edges:

$$N_{xy}(x=0) = N_{xy}(x=L/B) = N_{xy}(y=0) = N_{xy}(y=1) = 0. \tag{17}$$

Furthermore, the following in-plane conditions are enforced:

$$\int_0^1 N_x(x=0)\,dy = 0 \qquad \int_0^1 N_x(x=L/B)\,dy = 0 \tag{18}$$

$$\int_0^{L/B} N_y(y=0)\,dx = 0 \qquad \int_0^{L/B} N_y(y=1)\,dx = 0, \tag{19}$$

where $N_x = F_{,YY}$, $N_y = F_{,XX}$ and $N_{xy} = -F_{,XY}$. After some algebra, it can be shown that the above in-plane boundary conditions for a simply supported rectangular plate are equivalent to (see Koiter, [14], Section 63),

$$U(\text{at } x=0 \text{ or } x=L/B) = \text{constant}$$
$$V_{,X}(\text{at } x=0 \text{ or } x=L/B) = 0 \tag{20}$$

$$V(\text{at } y=0 \text{ or } y=1) = \text{constant}$$
$$U_{,Y}(\text{at } y=0 \text{ or } y=1) = 0, \tag{21}$$

where U and V are the in-plane axial and in-plane transverse displacements, respectively. It can be seen that $f_{11}(x,\,y)$ satisfies all the above boundary conditions.

In a similar manner, the second order field $f_{22}(x, y)$ can be written as,

$$f_{22}(x, y) = e_3 \cos(2M_2\pi x) + e_4 \cos(2\pi y), \tag{22}$$

where

$$e_3 = c/(16M_2^2)$$
$$e_4 = cM_2^2/16. \tag{23}$$

Finally, the differential equation for the second order field $f_{12}(x, y)$ is

$$f_{12,xxxx} + f_{12,yyyy} + 2f_{12,xxyy} = (c)[2w_{1,xy}w_{2,xy} - w_{1,xx}w_{2,yy} - w_{1,yy}w_{2,xx}], \tag{24}$$

and the solution for $f_{12}(x, y)$ can be written in closed form,

$$f_{12}(x, y) = c_1 \cos[(M_1 - M_2)\pi x] + c_2 \cos[(M_1 + M_2)\pi x] + \{c_3 \cos[(M_1 - M_2)\pi x] + c_4 \cos[(M_1 + M_2)\pi x]\} \cos(2\pi y), \tag{25}$$

where

$$c_1 = \frac{(-c/4)}{(M_1 - M_2)^2}, \qquad c_2 = \frac{(c/4)}{(M_1 + M_2)^2}$$
$$c_3 = \frac{(c/4)(M_1 + M_2)^2}{[(M_1 - M_2)^2 + 4]^2}, \quad c_4 = \frac{(-c/4)(M_1 - M_2)^2}{[(M_1 + M_2)^2 + 4]^2}. \tag{26}$$

4. POSTBUCKLING COEFFICIENTS

Using Budiansky–Hutchinson's notation [19] the equilibrium equations for the present two-mode problems are

$$b_1\xi_1^3 + b_{12}\xi_1\xi_2^2 + [1 - (\sigma/\sigma_1)]\xi_1 = (\sigma/\sigma_1)\bar{\xi}_1 \tag{27}$$

$$b_2\xi_2^3 + b_{21}\xi_1^2\xi_2 + [1 - (\sigma/\sigma_2)]\xi_2 = (\sigma/\sigma_2)\bar{\xi}_2, \tag{28}$$

where

$$b_1 = 4C_{40}/D_1, \quad b_2 = 4C_{40}/D_2$$
$$b_{12} = 2C_{22}/D_1, \quad b_{21} = 2C_{22}/D_2. \tag{29}$$

The C_{40}, C_{04}, C_{22}, D_1 and D_2 coefficients are computed by evaluating the appropriate integrals involving w_1, w_2, f_{11}, f_{22} and f_{12} (see Hui [15]). It follows that

$$C_{40} = (1/4)\int_0^1 \int_0^{L/B} [f_{11,yy}(w_{1,x})^2 + f_{11,xx}(w_{1,y})^2 - 2f_{11,xy}(w_{1,x})(w_{1,y})]\,dx\,dy$$
$$= (c\pi^4/128)(1 + M_1^4)(L/B) \tag{30}$$

and similarly,

$$C_{04} = (c\pi^4/128)(1 + M_2^4)(L/B). \tag{31}$$

Finally, the coefficient C_{22} can be obtained from

$$C_{22} = C_{22A} + C_{22B} + C_{22C}, \tag{32}$$

where

$$C_{22A} = (1/4)\int_0^1 \int_0^{L/B} f_{11,yy}(w_{2,x})^2 + f_{11,xx}(w_{2,y})^2 - 2f_{11,xy}(w_{2,x})(w_{2,y})\,dx\,dy$$
$$= (\pi^4/8)(e_2)(M_2^2) \tag{33}$$

and similarly

$$C_{22B} = (\pi^4/8)(e_4)(M_1^2). \tag{34}$$

Further, C_{22C} is

$$C_{22C} = \int_0^1 \int_0^{L/B} [f_{12,yy}(w_{1,x})(w_{2,x}) + f_{12,xx}(w_{1,y})(w_{2,y}) - f_{12,xy}(w_{1,x}w_{2,y} + w_{1,y}w_{2,x})]\,dx\,dy$$
$$= (\pi^4/8)\{(2M_1M_2)(c_3 + c_4) - (M_1 - M_2)^2[c_1 - (c_3/2)] + (M_1 + M_2)^2[c_2 - (c_4/2)]\}. \tag{35}$$

The quantities D_1 and D_2 are obtained from

$$D_1 = |f_{p,yy}| \int\int (w_{1,x})^2\,dx\,dy = [\sigma_1\pi^4M_1^2/(8c)](L/B) \tag{36}$$

$$D_2 = [\sigma_2\pi^4M_2^2/(8c)](L/B). \tag{37}$$

Finally, the equilibrium equation for a structure with a purely beneficial geometric imperfection is

$$b_2\xi_2^3 + [1 - (\sigma/\sigma_2)]\xi_2 = (\sigma/\sigma_2)\bar{\xi}_2 \tag{38}$$

and the stability boundary is

$$b_{12}\xi_2^2 + [1 - (\sigma_2/\sigma_1)(\sigma/\sigma_2)] = 0. \tag{39}$$

5. DISCUSSION

Figure 2 shows a graph of the buckling load vs the wave number M (defined to be mB/L). In the case where the aspect ratio of the rectangular plate L/B is a positive integer or when the plate is infinitely long ($L/B = \infty$), the

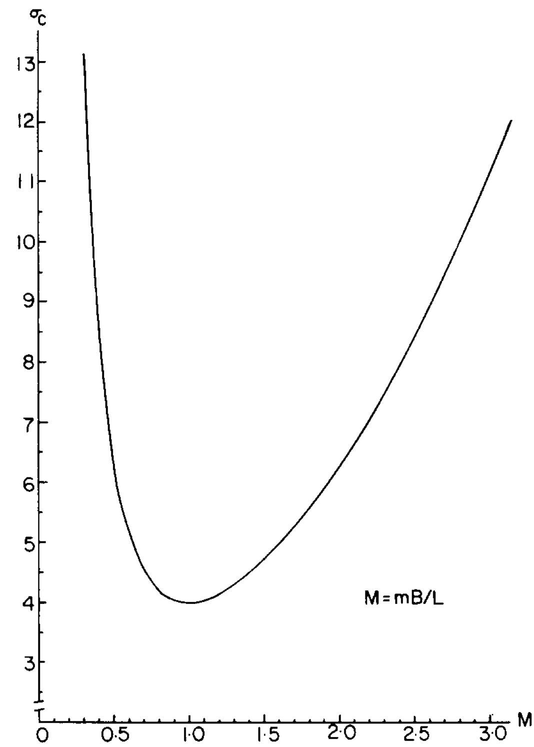

FIG. 2. Buckling load vs wave number for thin-walled square box columns under compression.

critical wave number is $M_c = 1$ and the buckling load is $\sigma_c = 4$. The following two cases of mode interaction are considered:

case one $\quad \sigma_1(M_1 = 1) = 4, \quad \sigma_2(M_2 = 2) = 6.25, \quad \sigma_1/\sigma_2 = 0.64 \quad (40)$

case two $\quad \sigma_1(M_1 = 1/\sqrt{2}) = 4.5, \quad \sigma_2(M_2 = \sqrt{2}) = 4.5, \quad L/B = \sqrt{2}. \quad (41)$

The b_2, b_{12} and b_{21} coefficients which are necessary to specify the postbuckling problem for $M_1 = 1$, $b_1 = 0.341250$ are plotted against M_2 in Fig. 3(a). Thus, for case one with $M_2 = 2$, the postbuckling coefficients are:

$$b_1 = 0.341250; \quad b_2 = 0.464099; \quad b_{12} = 1.918831; \quad b_{21} = 0.307013. \tag{42}$$

The b_{12} coefficient increases with M_2 while the b_{21} coefficient decreases with M_2 and these two curves intersect at $M_2 = M_1 = 1$. The b_2 coefficient decreases and then increases with the wave number M_2 and it has a minimum value of 0.341250 at $M_2 = 1$.

Similar curves of the b_2, b_{12} and b_{21} coefficients for $M_1 = 1/\sqrt{2} = 0.7071068$, $b_1 = 0.379167$ vs the wave number M_2 are presented in Fig. 3(b). Again the b_{12} and b_{21} curves intersect at $M_2 = M_1 = 0.7071068$. For case two, with $M_2 = \sqrt{2} = 1.414213$, the postbuckling coefficients are found to be:

$$b_1 = b_2 = 0.379167; \quad b_{12} = 1.821048; \quad b_{21} = 0.455262. \tag{43}$$

Note that the above b coefficients for $L/B = \sqrt{2}$ correspond to the Compact-plus double-cusp model [20, 21] and they disagree (due to possible analytical errors) with the result reported by Magnus and Poston [22].

The equilibrium paths for case one [specified by equations (38) and (39)] for the structure with a purely beneficial geometric imperfection (with $\bar{\xi}_1 = 0$) are plotted in Fig. 4(a). A sign change in the amplitude ξ_2 implies a sign change in the beneficial imperfection amplitude $\bar{\xi}_2$ so that it is not necessary to plot negative values of the amplitude ξ_2. For small values of the beneficial imperfection amplitude ($\bar{\xi}_2 < 0.118$), the equilibrium path intersects the stability boundary at two points. However, for larger values of $\bar{\xi}_2 > 0.118$, the equilibrium path does not intersect the stability boundary and thus, the structure will not bifurcate into the w_1 mode as the applied axial load is increased. Such transitional imperfection amplitude of the beneficial mode versus the wave number M_2 are plotted in Fig. 5.

Similar equilibrium paths and the stability boundary for case two are plotted in Fig. 4(b). However, none of the equilibrium paths of the imperfect system intersect the stability boundary. This means that regardless of how small the amplitude of the beneficial geometric imperfection (provided $\bar{\xi}_2 \neq 0$), the equilibrium path will not bifurcate into the w_1 mode. The equilibrium paths for other cases can be plotted and it is expected that they will qualitatively behave as in either Fig. 4(a) or (b).

Finally, in the case where both geometric imperfections are nonzero ($\bar{\xi}_1 \neq 0, \bar{\xi}_2 \neq 0$), the equilibrium paths can be obtained by solving simultaneously equations (27) and (28) for given values of imperfection amplitudes $\bar{\xi}_1$, $\bar{\xi}_2$ and applied load σ/σ_2. However, it is much simplier if values of $\bar{\xi}_1$, $\bar{\xi}_2$ and amplitude of the beneficial mode ξ_2 are given, treating σ/σ_1 and ξ_1 as the two unknown variables. Re-arranging equation (28), one obtains,

$$\sigma/\sigma_2 = k_1 + (k_2)(\xi_1)^2, \tag{44}$$

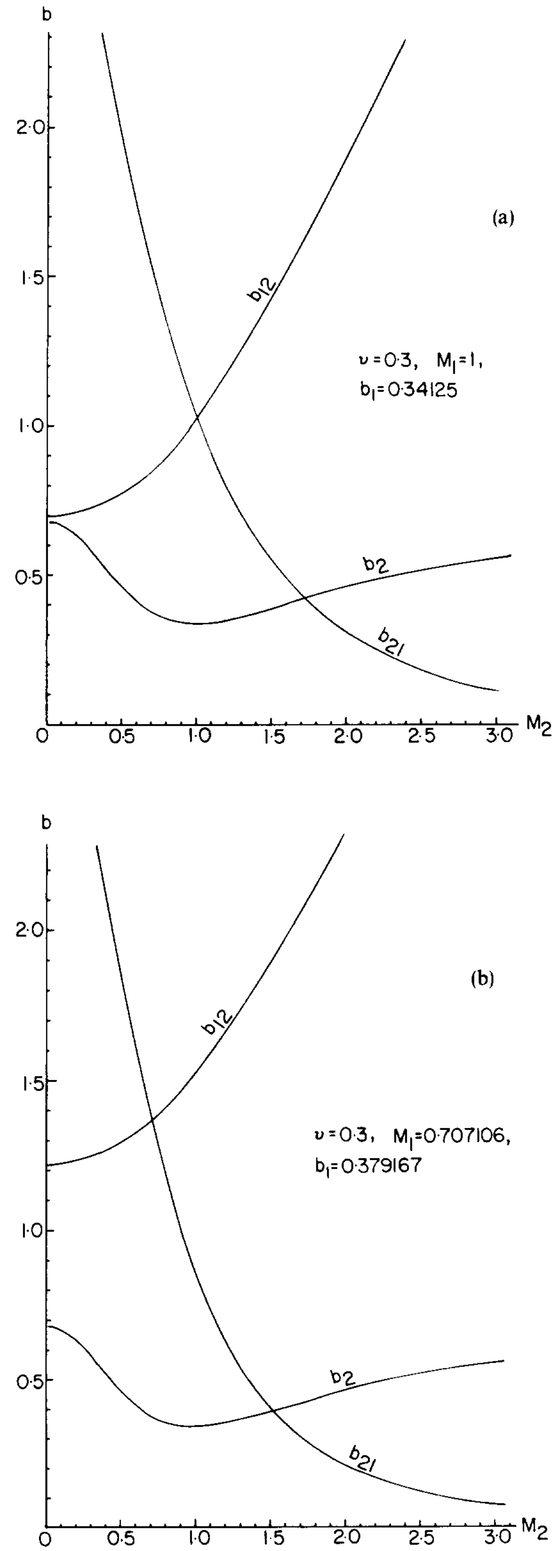

FIG. 3. (a) The b_2, b_{12} and b_{21} coefficients vs wave number M_2 (with $\nu = 0.3$, $M_1 = 1$ and $b_1 = 0.341250$); (b) the b_2, b_{12} and b_{21} coefficients vs wave number M_2 (with $\nu = 0.3$, $M_1 = 1/\sqrt{2}$ and $b_1 = 0.379167$).

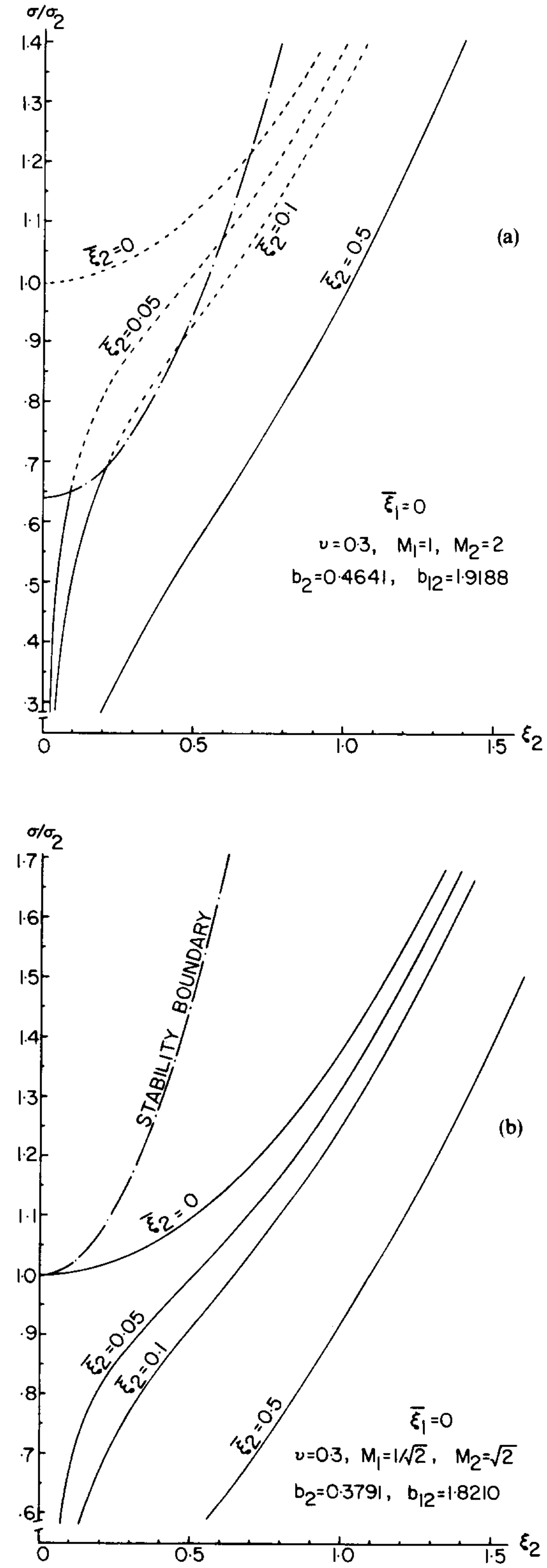

FIG. 4. (a) Equilibrium paths of thin-walled box column under compression with purely beneficial imperfections ($\nu = 0.3$, $M_1 = 1$, $M_2 = 2$); (b) equilibrium paths of thin-walled box columns under compression with purely beneficial imperfections ($\nu = 0.3$, $M_1 = 0.707106$, $M_2 = 1.414213$).

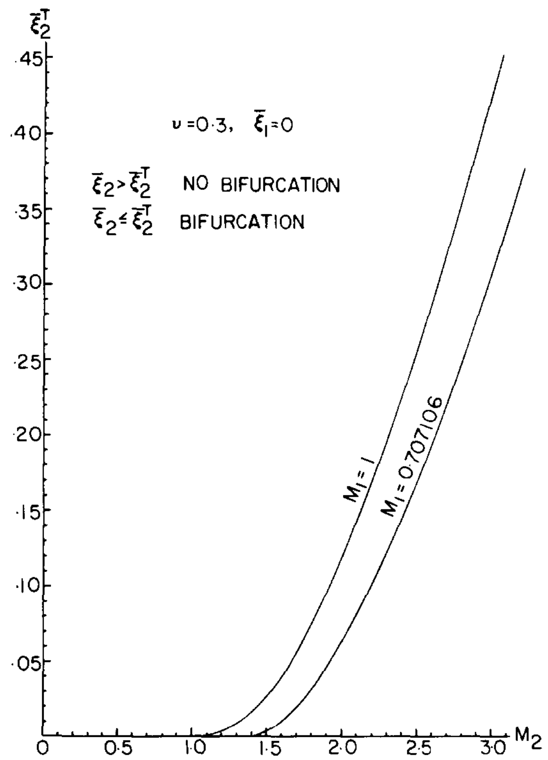

FIG. 5. Transitional value of the beneficial imperfection amplitude vs wave number of the beneficial mode ($\nu = 0.3$, $\bar{\xi}_1 = 0$).

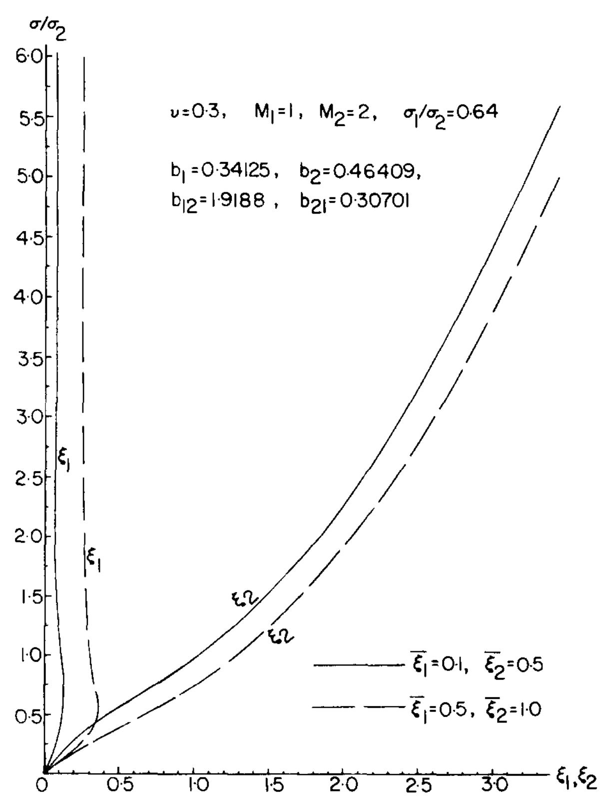

FIG. 6. Equilibrium paths of thin-walled box columns under compression with combined imperfections (Case one, $\nu = 0.3$, $M_1 = 1$, $M_2 = 2$, $\sigma_1/\sigma_2 = 0.64$).

where

$$k_1 = [(b_2)(\xi_2)^3 + \xi_2]/(\xi_2 + \bar{\xi}_2), \qquad k_2 = b_{21}\xi_2/(\xi_2 + \bar{\xi}_2). \tag{45}$$

Substituting equation (44) into equation (27), one obtains a cubic equation in ξ_1 in the form,

$$[b_1 - (k_2)(\sigma_2/\sigma_1)](\xi_1)^3 - (k_2\bar{\xi}_1)(\sigma_2/\sigma_1)(\xi_1)^2 + [1 + b_{12}(\xi_2)^2 - (k_1)(\sigma_2/\sigma_1)](\xi_1) - (k_1\bar{\xi}_1)(\sigma_2/\sigma_1) = 0. \tag{46}$$

Thus, for fixed values of the imperfection amplitudes $\bar{\xi}_1$, $\bar{\xi}_2$, the load ratio σ_2/σ_1 and the b coefficients, one can solve for the amplitude ξ_1 for given values of ξ_2. The applied load σ/σ_2 can then be obtained from equation (44).

The equilibrium paths for case one [equation (40)] are plotted in Fig. 6 for values of the imperfection amplitudes being ($\bar{\xi}_1 = 0.1$, $\bar{\xi}_2 = 0.5$) and ($\bar{\xi}_1 = 0.5$, $\bar{\xi}_2 = 1.0$). It can be seen that for combined imperfections, provided the amplitude of the beneficial imperfection is sufficiently large, the amplitude of the beneficial mode increases monotonically with the applied load. The amplitude of the less desirable mode approaches a small asymptotic value as the applied load is increased. Thus, it is possible to design a thin-walled box column to collapse in the beneficial mode, even if there is an unavoidable imperfection in the shape of the less desirable mode.

6. CONCLUDING REMARKS

The effects of beneficial geometric imperfections on the elastic collapse behavior of thin-walled square box columns have been examined. Based on the two-mode potential expression, the equilibrium paths of the columns with purely beneficial geometric imperfections are plotted. It is found that it is possible to design a built-in beneficial imperfection to achieve high elastic post-collapse stiffness in the finite-deflection regime. The present analysis is valid only for sufficiently small values of the imperfection amplitude.

The same method of analysis could be of invaluable assistance in designing members subjected to bending collapse where only one fold is expected to be formed [23]. Further work on the effects of beneficial imperfections on bending collapse of rectangular and square tubes [24] will be presented in a subsequent paper.

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REFERENCES

1. S. P. Timoshenko, Theory of bending, torsion and buckling of thin-walled members of open cross section. *J. Franklin Inst.* **239**, 201–219, 249–268 and 343–361; also published in *The Collected Papers of Stephen P. Timoshenko*. pp. 559–609. McGraw-Hill, New York (1953).
2. T. R. Graves Smith, and S. Sridharan, The local collapse of elastic thin walled columns. *J. struct. Mech.* **8**, 471–489 (1980).
3. T. Wierzbicki and T. Akerstrom, Dynamic crushing of strain rate sensitive box columns. *Proc. 2nd Int. Conf. Vehicle Struct. Mech.* (*VSM*), Southfield, Michigan. 18–20 April, 1977 pp. 19–31. Published by Society of Automotive Engineers (1977).
4. T. Wierzbicki, On the formation and growth of folding modes. *Collapse: The Buckling of Structures in Theory and Practice, IUTAM Symp.* (Edited by J. M. T. Thompson and G. W. Hunt), pp. 19–30, 31 August–3 Sept., 1982. Cambridge University Press (1983).
5. T. Wierzbicki, Crushing behaviour of plate intersections. *Symp. Structural Crashworthiness*, University of Liverpool (Edited by N. Jones and T. Wierzbicki), pp. 66–95. Butterworths, London (1983).
6. C. L. Magee and P. H. Thornton, Design considerations in energy absorption by structural collapse. *Soc. automot. Engrs* **87**, 2041–2055. SAE Paper No. 780434 (1978).
7. W. Abramowicz and N. Jones, Dynamic axial crushing of square tubes. *Int. J. Impact Engng* **2**, 179–208 (1984).
8. Q. Meng, S. T. S. Al-Hassani and P. D. Soden, Axial crushing of square tubes. *Int. J. Mech. Sci.* **25**, 747–773 (1983), presented at Symposium on Structural Crashworthiness, University of Liverpool, 14–16 Sept. (1983).
9. T. Usami, and Y. Fukumoto, Local and overall buckling of welded box columns. *ASCE J. Struct. Div.* **108**, No. ST3, 525–542 (1982).
10. B. Rawlings and P. Shapland, The behaviour of thin-walled box sections under gross deformation. *Struct. Engr* **53**, 181–186 (1975).
11. H. F. Mahmood and A. Paluszny, Design of thin walled columns for crash energy management—Their strength and mode of collapse. *Proc. 4th Int. Conf. Vehicle Struct. Mech.* (VSM), Detroit, Michigan. 18–20 Nov., 1981, pp. 7–18. Published by Society of Automotive Engineers (1981).
12. N. W. Murray, The static approach to plastic collapse and energy dissipation in some thin-walled steel structures. *Symp. Structural Crashworthiness*, (Edited by N. Jones and T. Wierzbicki), pp. 44–65. University of Liverpool Butterworths, London (1983).
13. K. Murase and H. Katori, Visco-plastic large deformation analysis by finite element method and its applications. *Bull. J.S.M.E.* **26**, 923–929 (1983).
14. W. T. Koiter, On the stability of elastic equilibrium. Doctoral thesis, Delft, Amsterdam; English translations NASA-TT-F10, 833 (1967) and AFFDLTR-70-25 (1970).
15. D. Hui, Effects of mode interaction on collapse of short, imperfect, thin walled columns. *ASME J. appl Mech.* **51**, 566–573 (1984).
16. D. Hui and A. W. Leissa, Effect of geometric imperfections on vibrations of biaxially compressed rectangular flat plates. *ASME J. appl. Mech.* **50**, 750–756 (1983).
17. D. Hui and J. S. Hansen, Two-mode buckling of an elastically supported plate and its relation to catastrophe theory. *ASME J. appl. Mech.* **47**, 607–612 (1980).

18. S. P. TIMOSHENKO and J. M. GERE, *Theory of Elastic Stability*. McGraw-Hill, New York (1961).
19. B. BUDIANSKY and J. W. HUTCHINSON, Dynamic buckling of imperfection-sensitive structures. *Proc. 11th Int. Congr. appl. Mech.* 1964 (Edited by H. GORTLER), pp. 634–651. Springer-Verlag, Berlin (1966).
20. J. CALLAHAN, Special bifurcations of the double cusp. *Proc. Lond. math. Soc.* Series 3, **45**, 227–257 (1982).
21. D. HUI, Imperfection sensitivity of elastically supported beams and its relation to the double-cusp instability model. *Proc. R. Soc. Lond.* Series A. (In press.)
22. R. MAGNUS and T. POSTON, On the full unfolding of the Von Kármán equations at a double eigenvalue, Battelle Laboratory, (Switzerland). Mathematics Report No. 109 (August 1977).
23. T. WIERZBICKI and S. U. BHAT, A moving hinge solution for axisymmetric crushing of tubes. M.I.T. Dept of Ocean Engrg. Report No. 85-3 (February 1985).
24. D. KECMAN, Bending collapse of rectangular and square section tubes. *Int. J. Mech. Sci.* **25**, 623–636 (1983).