Initial Postbuckling Behavior of Imperfect, Antisymmetric Cross-Ply Cylindrical Shells Under Torsion

David Hui
Assistant Professor,
Mem. ASME

I. H. Y. Du
Ph.D. Student.

The Ohio State University,
Department of Engineering Mechanics,
Columbus, Ohio 43210

This paper deals with the initial postbuckling of antisymmetric cross-ply closed cylindrical shells under torsion. Under the assumptions employed in Koiter's theory of elastic stability, the structure is imperfection-sensitive in certain intermediate ranges of the reduced-Batdorf parameter (approx. $4 \leq Z_H \leq 20.0$). Due to different material bending-stretching coupling behavior, the (0 deg inside, 90 deg outside) two-layer clamped cylinder is less imperfection sensitive than the (90 deg inside, 0 deg outside) configuration. The increase in torsional buckling load due to a higher value of Young's moduli ratio is not necessarily accompanied by a higher degree of imperfection-sensitivity. The paper is the first to consider imperfection shape to be identical to the torsional buckling mode and presents concise parameter variations involving the reduced-Batdorf paramter and Young's moduli ratio.

1 Introduction

Due to increased use of light-weight, high-strength composite materials in aerospace, mechanical, and hydrospace industries, buckling and postbuckling of laminated closed cylindrical shells has become an important research topic. One of the earliest stability analyses of orthotropic cylindrical shells under torsion was presented by March et al. (1945). Subsequently, Hayashi and Hirano (1964), Chechill and Cheng (1968), Shaw and Simitses (1982, 1984) and Simitses et al. (1985) gave further examinations of this problem. Comprehensive experimental investigations of torsional buckling of laminated cylindrical shells were presented by Marlowe et al. (1973), Wilkins and Love (1974), Booton (1976), Booton and Tennyson (1979), and Herakovich and Johnson (1981). A detailed summary of the literature on torsional buckling of laminated cylindrical shells can be found in review articles by Tennyson (1975) and Simitses (1986). In passing, comprehensive reviews on buckling of composite cylindrical shells under compression and external pressure were presented by Tennyson (1984) and Galletly and Pemsing (1985). An excellent review on buckling of laminated open cylindrical panels was reported by Leissa (1985) and three recent texts were written by Kollar and Dulacska (1984), Yamaki (1984), and Bushnell (1985).

The present work was motivated by the work of Budiansky (1967), Booton (1976), and Simitses, Shaw, and Sheinman (1985). The limitations of these works will be addressed in this paper. Budiansky (1967) presented a rigorous initial postbuckling behavior of isotropic homogeneous cylindrical shells under torsion, using Koiter's (1945) general stability theory, but he did not consider laminated cylinders. Booton (1976) considered the effects of "axisymmetric" geometric imperfections on torsional or combined-load buckling of laminated cylindrical shells, using Koiter's (1963) special stability theory, but he did not consider nonaxisymmetric imperfections or postbuckling. Simitses, Shaw, and Sheinman (1985) presented a rigorous large deflection analysis of laminated cylinders under torsion. The imperfection terms were present in the governing equations and they were solved using a generalization of Newton's method applicable to differential equations (Thurston, 1965). They employed imperfection shapes which are "similar" to the buckling modes of the geometry as well as to other shapes. They obtained these shapes by employing the methodology of Shaw and Simitses (1984). The present study attempts to examine in detail problems similar to the one examined by Simitses et al. using Koiter's well-accepted and equally rigorous alternative approach. Particular emphasis is placed on the "nonaxisymmetric" imperfection shapes which are identical to those of the buckling modes. These shapes are likely to be more realistic than the axisymmetric imperfections.

The present paper is an extension of Budiansky (1967) and Yamaki and Kodama's (1980) work to "laminated" cylindrical shells. The analysis is based on a solution of the Donnell-type nonlinear equilibrium and compatibility equations, using Koiter's (1945) general theory of elastic stability. Using a Koiter-type perturbation method, a sequence of linearized differential equations is derived. The classical

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buckling load and the corresponding buckling mode are obtained by discretizing the governing differential equations and boundary conditions using the central finite difference scheme. The resulting eigenvalue problem is solved using the shifted inverse power method in conjunction with the LINPACK equation solver (Dongarra et al., 1979). The cubic terms of the potential energy vanish due to symmetry considerations and computation of the quartic term (related to the postbuckling b coefficients defined by Budiansky, 1967) requires solution of the second order field equations. It should be noted that the present Koiter-type analysis is valid only for imperfection shapes identical to that of the buckling mode and for imperfection amplitudes which are sufficiently small.

The example problems are chosen for antisymmetric cross-ply cylindrical shells under torsion. The shells are either clamped or simply-supported and the in-plane boundary conditions are axially and circumferentially immovable in the buckling state and second order field solution. Due to material bending-stretching coupling, the two layer (0 deg inside, 90 deg outside) and the (90 deg, 0 deg) designs have almost identical torsional buckling load but different postbuckling stiffness. The parameter variations involving the reduced-Batdorf parameter and Young's moduli ratio are presented.

2 Governing Nonlinear Differential Equations and the Classical Buckling Load

The initial postbuckling behavior of geometrically imperfect, laminated cylindrical shells is based on a solution of the nonlinear Donnell-type equilibrium and compatibility equations, written in terms of an out-of-plane displacement W (positive outwards) and a stress function F, respectively (Hui 1985a,b).

$$L_D^*(W) + L_B^*(F) + (1/R)F_{,XX} = F_{,YY}W_{,XX} + F_{,XX}W_{,YY} - 2F_{,XY}W_{,XY} \quad (1)$$

$$L_A^*(F) - L_B^*(W) - (1/R)W_{,XX} = (W_{,XY})^2 - W_{,XX}W_{,YY} \quad (2)$$

In the above R is the shell radius, X and Y are the axial and circumferential coordinates, and $L_A^*(\ )$, $L_B^*(\ )$, and $L_D^*(\ )$ are the linear fourth order differential operators defined by Tennyson et al. (1971, 1980).

Consistent with Koiter's (1945) theory of elastic stability, the total displacement and the total stress fucntion can be expressed as the sum of the prebuckling state, the buckling state, and the second order field in the form (Hui et al., 1981)

$$w = w_p + \xi w_I + \xi^2 w_{II}, \qquad f = f_p + \xi f_I + \xi^2 f_{II} \quad (3)$$

In the above, ξ is the amplitude of the buckling mode normalized with respect to the total laminated thickness h, $w = W/h$, $f = F/(E_2h^3)$ and E_2 is Young's modulus in the in-plane transverse direction. The nondimensional applied axial load σ, the lateral pressure p, and the nondimensional torsional load τ are related to the prebuckling stress function and the membrane stress resultants by,

$$(\sigma,p,\tau) = (f_{p,yy}, f_{p,xx}, -f_{p,xy}) = (|N_x|, N_y, N_{xy})[R/(E_2h^2)] \quad (4)$$

Assuming a membrane prebuckling state and introducing the following nondimensional quantities,

$$(x,y) = (X,\ Y)/(Rh)^{1/2}, \quad a_{ij}^* = (E_2h)A_{ij}^*,\ b_{ij}^* = B_{ij}^*/h, \quad d_{ij}^* = D_{ij}^*/(E_2h^3) \quad (5)$$

the nondimensional linearized equilibrium and compatibility equations for the buckling state becomes,

$$L_d^*(w_I) + L_b^*(f_I) + f_{I,xx} + \sigma w_{I,xx} + p w_{I,yy} - (2\tau) w_{I,xy} = 0 \quad (6)$$

$$L_a^*(f_I) - L_b^*(w_I) - w_{I,xx} = 0 \quad (7)$$

The nondimensional differential operators are defined as Hui (1986a),

$$L_a^*(\ ) = a_{22}^*(\ )_{,xxxx} + (2a_{12}^* + a_{66}^*)(\ )_{,xxyy} + a_{11}^*(\ )_{,yyyy} - 2a_{26}^*(\ )_{,xxxy} - 2a_{16}^*(\ )_{,xyyy} \quad (8)$$

$$L_b^*(\ ) = b_{21}^*(\ )_{,xxxx} + (b_{11}^* + b_{22}^* - 2b_{66}^*)(\ )_{,xxyy} + b_{12}^*(\ )_{,yyyy} + (2b_{26}^* - b_{61}^*)(\ )_{,xxxy} + (2b_{16}^* - b_{62}^*)(\ )_{,xyyy} \quad (9)$$

$$L_d^*(\ ) = d_{11}^*(\ )_{,xxxx} + (2)(d_{12}^* + 2d_{66}^*)(\ )_{,xxyy} + d_{22}^*(\ )_{,yyyy} + 4d_{16}^*(\ )_{,xxxy} + 4d_{26}^*(\ )_{,xyyy} \quad (10)$$

The general solution for buckling of a laminated cylindrical shell under pure torsion or combined load involving torsion is,

$$w_I(x,y) = w_s(x)\sin(Ny) + w_c(x)\cos(Ny), \qquad f_I(x,y) = f_s(x)\sin(Ny) + f_c(x)\cos(Ny) \quad (11)$$

where $N = n(h/R)^{1/2}$. Substituting $w_I(x, y)$ and $f_I(x, y)$ into the equilibrium equation for the buckling state and collecting terms involving $\sin(Ny)$ and $\cos(Ny)$, one obtains two ordinary differential equations, respectively,

$$L_d^{**}[w_s(x)] - L_d^{***}[w_c(x)] + L_b^{**}[f_s(x)] - L_b^{***}[f_c(x)] + f_s(x)_{,xx} + \sigma w_s(x)_{,xx} - pN^2 w_s(x) + (2\tau)Nw_c(x)_{,x} = 0 \quad (12)$$

$$L_d^{**}[w_c(x)] + L_d^{***}[w_s(x)] + L_b^{**}[f_c(x)] + L_b^{***}[f_s(x)] + f_c(x)_{,xx} + \sigma w_c(x)_{,xx} - pN^2 w_c(x) - (2\tau)Nw_s(x)_{,x} = 0 \quad (13)$$

Similarly, collecting $\sin(Ny)$ and $\cos(Ny)$ from the compatibility equation for the buckling state, one obtains, respectively,

$$L_a^{**}[f_s(x)] - L_a^{***}[f_c(x)] - L_b^{**}[w_s(x)] + L_b^{***}[w_c(x)] - w_s(x)_{,xx} = 0 \quad (14)$$

$$L_a^{**}[f_c(x)] + L_a^{***}[f_s(x)] - L_b^{**}[w_c(x)] - L_b^{***}[w_s(x)] - w_c(x)_{,xx} = 0 \quad (15)$$

In the above equations, the ordinary differential operators are defined as

$$L_a^{**}(\ ) = a_{22}^*(\ )_{,xxxx} - N^2(2a_{12}^* + a_{66}^*)(\ )_{,xx} + N^4 a_{11}^*(\ ), \qquad L_a^{***}(\ ) = -2a_{26}^*N(\ )_{,xxx} + 2a_{16}^*N^3(\ )_{,x} \quad (16a,b)$$

$$L_b^{**}(\ ) = b_{21}^*(\ )_{,xxxx} - N^2(b_{11}^* + b_{22}^* - 2b_{66}^*)(\ )_{,xx} + N^4 b_{12}^*(\ ), \qquad L_b^{***}(\ ) = (2b_{26}^* - b_{61}^*)N(\ )_{,xxx} - (2b_{16}^* - b_{62}^*)N^3(\ )_{,x} \quad (17a,b)$$

$$L_d^{**}(\ ) = d_{11}^*(\ )_{,xxxx} - 2N^2(d_{12}^* + 2d_{66}^*)(\ )_{,xx} + N^4 d_{22}^*(\ ), \qquad L_d^{***}(\ ) = 4d_{16}^*N(\ )_{,xxx} - 4d_{26}^*N^3(\ )_{,x} \quad (18a,b)$$

The boundary conditions considered in this paper are assumed to be identical at both ends of the cylindrical shells at $x = 0$ and at $x = L/(Rh)^{1/2}$ and $w(x=0) = 0$, $w(x = L/(Rh)^{1/2}) = 0$. The out-of-plane boundary conditions are either clamped $w_{,x}$ $(x = 0) = 0$ or simply-supported M_x $(x = 0) = 0$. In the mixed formulation in terms of w and f, the zero bending moment condition is (Hui, 1986a)

$$-b_{11}^* f_{,yy}(x=0) - b_{21}^* f_{,xx}(x=0) + b_{61}^* f_{,xy}(x=0) - d_{11}^* w_{,xx}(x=0) - 2d_{16}^* w_{,xy}(x=0) = 0 \quad (19)$$

Note that $(\ ) = 0$ at $x = 0$ implies $(\ )_{,y} = 0$ and $(\ )_{,yy} = 0$, etc. at $x = 0$. The in-plane boundary conditions are either axially immovable or axially movable, that is, (Booton, 1976),

$$U_{,YY}(x=0) = 0 \quad \text{or} \quad f_{,yy}(x=0) = 0 \quad (20)$$

Further, the shell is either circumferentially immovable or circumferentially movable,

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$$V,_Y(x=0)=0 \quad \text{or} \quad f,_{xy}(x=0)=0 \tag{21}$$

Using the following strain-displacement relation at $x = 0$,

$$-\epsilon_{y,X}(X=0)+\gamma_{xy,Y}(X=0) = U,_{YY}(X=0)-(1/R)W,_X(X=0) \tag{22}$$

the axially immovable boundary condition can be written in terms of w and f,

$$[a^*_{16}f,_{yyy}+2a^*_{26}f,_{xxy}-(a^*_{12}+a^*_{66})f,_{xyy} - a^*_{22}f,_{xxx}-0+(b^*_{22}-2b^*_{66})w,_{xyy} + (2b^*_{26}-b^*_{61})w,_{xxy}+b^*_{21}w,_{xxx}+w,_x]=0 \quad \text{at } x=0 \tag{23}$$

Note that $\epsilon_y = V,_{Y+}\ (W/R) + (1/2)(W,_Y)^2$ and

$$\epsilon_y=(h/R)[a^*_{12}f,_{yy}+a^*_{22}f,_{xx} - a^*_{26}f,_{xy}-b^*_{21}w,_{xx}-b^*_{22}w,_{yy}-2b^*_{26}w,_{xy}] \tag{24a}$$

Using the strain-displacement condition that $\epsilon_y\ (X = 0) = V,_Y(X = 0)$, the circumferentially immovable boundary condition $V,_Y(X = 0) = 0$ becomes,

$$[a^*_{12}f,_{yy}+a^*_{22}f,_{xx}-a^*_{26}f,_{xy}-b^*_{21}w,_{xx}-2b^*_{26}w,_{xy}]=0 \quad \text{at } x=0 \tag{24b}$$

using the conditions $w(x = 0) = 0$, $w,_y(x = 0) = 0$ and $w,_{yy}(x = 0) = 0$.

In the present paper, the in-plane boundary conditions are $U,_{YY}(x = 0) = 0$ and $V,_Y(x = 0) = 0$ for the buckling and second order field analyses, while the prebuckling state is assumed to be a membrane one. Substituting the separable form of the buckling mode into the $U,_{YY}(X = 0) = 0$ boundary condition, and collecting terms involving sin(Ny) and cos(Ny), one obtains, respectively,

$$[a^*_{16}N^3f_c-2a^*_{26}Nf_{c,xx}+(a^*_{12}+a^*_{66})N^2f_{s,x} - a^*_{22}f_{s,xxx}-N^2(b^*_{22}-2b^*_{66})w_{s,x}-N(2b^*_{26} - b^*_{61})w_{c,xx}+b^*_{21}w,_{xxx}+w_{s,x}]=0 \quad \text{at } x=0 \tag{25}$$

$$[-N^3a^*_{16}f_s+2a^*_{26}Nf_{s,xx}+N^2(a^*_{12}+a^*_{66})f_{c,x} - a^*_{22}f_{c,xxx}-N^2(b^*_{22}-2b^*_{66})w_{c,x}+N(2b^*_{26} - b^*_{61})w_{s,xx}+b^*_{21}w_{c,xxx}+w_{c,x}]=0 \quad \text{at } x=0 \tag{26}$$

Similarly, collecting terms involving sin(Ny) and cos(Ny) in the $V,_Y(x = 0) = 0$ boundary condition, one obtains, respectively,

$$[-N^2a^*_{12}f_s+a^*_{22}f_{s,xx}+Na^*_{26}f_{c,x} - b^*_{21}w_{s,xx}+2b^*_{26}Nw_{c,x}]=0 \quad \text{at } x=0 \tag{27}$$

$$[-N^2a^*_{12}f_c+a^*_{22}f_{c,xx}+Na^*_{26}f_{s,x} - b^*_{21}w_{c,xx}-2b^*_{26}Nw_{s,x}]=0 \quad \text{at } x=0 \tag{28}$$

3 Second Order Fields and Postbuckling Coefficients

Due to the symmetry consideration that a sign change of the buckling mode represents a configuration which is identical to the original mode, the cubic term of the potential energy vanishes. In order to compute the quartic terms of the potential energy (Koiter, 1945), it is necessary to solve for the second order fields $w_{II}(x, y)$ and $f_{II}(x, y)$.

Substituting the total displacement and the total stress function (equation (3)) into the governing nonlinear equilibrium and compatibility equations and then collecting terms involving ξ^2, one obtains the two governing partial differential equations for the second order fields,

$$L_{d^*}(w_{II})+L_{b^*}(f_{II})+f_{II,xx}+\sigma w_{II,xx}+pw_{II,yy}-(2\tau)w_{II,xy} = f_{I,yy}w_{I,xx}+f_{I,xx}w_{I,yy}-2f_{I,xy}w_{I,xy} \tag{29}$$

$$L_{a^*}(f_{II})-L_{b^*}(w_{II})-w_{II,xx}=(w_{I,xy})^2-w_{I,xx}w_{I,yy} \tag{30}$$

Substituting the buckling mode $w_I(x, y)$ and $f_I(x, y)$ into the right-hand sides of equations (19) and (20), it is clear that the second order fields are of the form,

$$w_{II}(x,y) = w^*(x) + w_A(x)\cos(2Ny) + w_B(x)\sin(2Ny) \tag{31}$$

$$f_{II}(x,y) = f^*(x) + f_A(x)\cos(2Ny) + f_B(x)\sin(2Ny) \tag{32}$$

Thus, the equilibrium and compatibility equations for the second order fields may be uncoupled into two sets of ordinary differential equations. The first set involves $w^*(x)$ and $f^*(x)$ and the second set deals with $w_A(x)$, $w_B(x)$, $f_A(x)$ and $f_B(x)$. The first set is,

$$a_{22}f^*(x),_{xxxx}-b_{21}w^*(x),_{xxxx}-w^*(x),_{xx} = (N^2/4)[w_s(x)^2+w_c(x)^2],_{xx} \tag{33}$$

$$d^*_{11}w^*(x),_{xxxx}+b^*_{21}f^*(x),_{xxxx}+f^*(x),_{xx}+\sigma w^*(x),_{xx} = (-N^2/2)[w_s(x)f_s(x)+w_c(x)f_c(x)],_{xx} \tag{34}$$

It can be easily checked from equations (24a,b) that the singlevaluedness requirement of the circumferential displacement for the second order problem is precisely equation (33).

The zero-moment boundary condition at $x = 0$ for the $w^*(x)$ and $f^*(x)$ problem is (see equation (19)),

$$-b^*_{21}f^*_{xx}(x=0)-d^*_{11}w,^*_{xx}(x=0)=0 \tag{35}$$

Further, the $U,_{YY}(x = 0)$ condition for the y-independent $w^*(x)$ and $f^*(x)$ problem is (see equation (23)),

$$-a^*_{22}f,^*_{xxx}(x=0)+b^*_{21}w,^*_{xxx}(x=0)+w^*_{,x}(x=0)=0 \tag{36}$$

and similarly, the $V,_Y(x = 0) = 0$ condition can be obtained from equation (24),

$$a^*_{22}f,^*_{xx}(x=0)-b^*_{21}w,^*_{xx}(x=0)=0 \tag{37}$$

The second set of four coupled ordinary differential equations are obtained from the compatibility equation which involved cos (2Ny) and sin (2Ny), respectively,

$$L_{a^{**}}[f_A(x)]+L_{a^{***}}[f_B(x)]-L_{b^{**}}[w_A(x)] - L_{b^{***}}[w_B(x)]-w_A(x),_{xx}=(N^2/2)[(w_{s,x})^2 - (w_{c,x})^2-w_sw_{s,xx}+w_cw_{c,xx}] \tag{38}$$

$$L_{a^{**}}[f_B(x)]-L_{a^{***}}[f_A(x)]-L_{b^{**}}[w_B(x)] + L_{b^{***}}[w_A(x)]-w_B(x),_{xx}=(N^2/2)[(-2)(w_{s,x})(w_{c,x}) + w_cw_{s,xx}+w_sw_{c,xx}] \tag{39}$$

and from the equilibrium equation which involves cos (2Ny) and sin (2Ny), respectively,

$$L_{d^{**}}[w_A(x)]+L_{d^{***}}[w_B(x)]+L_{b^{**}}[f_A(x)]+L_{b^{***}}[f_B(x)] +f_A(x),_{xx}+(\sigma)w_A(x),_{xx}-(2N)^2pw_A(x)-(4N)(\tau)w_B(x),_x = (N^2/2)[w_{s,xx}f_s-w_{c,xx}f_c+w_sf_{s,xx}-w_cf_{c,xx} -2w_{s,x}f_{s,x}+2w_{c,x}f_{c,x}] \tag{40}$$

$$L_{d^{**}}[w_B(x)]-L_{d^{***}}[w_A(x)]+L_{b^{**}}[f_B(x)]-L_{b^{***}}[f_A(x)] +f_B(x),_{xx}+(\sigma)w_B(x),_{xx}-(2N)^2pw_B(x)+(4N)(\tau)w_A(x),_x = (N^2/2)[-w_{c,xx}f_s-w_{s,xx}f_c-w_cf_{s,xx}-w_sf_{c,xx} +2w_{c,x}f_{s,x}+2w_{s,x}f_{c,x}] \tag{41}$$

It is important to note that the linear differential operators $L_{a^{**}}(\ )$, $L_{a^{***}}(\ )$, $L_{b^{**}}(\ )$, $L_{b^{***}}(\ )$, $L_{d^{**}}(\ )$, and $L_{d^{***}}(\ )$ employed in equations (38)–(41) are the same as that presented for the buckling state except that the wave number N should be replaced by 2N in equations (16a,b), (17a,b), and (18a,b).

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Table 1 Torsional buckling load and b coefficient for Boron-epoxy cross-ply cylindrical shells

	(i) Clamped N_L = infinite			(ii) Clamped (0° out, 90° in)		(iii) Clamped (90° in, 0° out)		(iv) s.s N_L = infinite	
Z_H	N	τ_c	b	N	τ_c	N	τ_c	N	τ_c
1.0	3.82	30.51	0.476	3.62	16.53	3.66	16.79	2.42	17.90
2.0	2.28	7.619	0.392	2.02	4.157	2.12	4.183	1.68	4.183
3.0	1.50	3.492	0.114	1.44	1.975	1.42	1.980	1.10	2.301
4.0	1.22	2.143	0.0409	1.04	1.357	1.00	1.356	1.08	1.515
5.0	1.10	1.516	−0.00170	1.00	0.9713	0.98	0.9623	1.00	1.174
6.0	1.02	1.199	−0.0193	1.00	0.7599	0.98	0.7570	0.90	0.9674
7.0	0.96	0.9890	−0.0308	1.02	0.6500	0.98	0.6493	0.86	0.8653
8.0	0.92	0.8854	−0.0316	0.98	0.5863	0.96	0.5826	0.88	0.7905
9.0	0.86	0.8005	−0.0333	0.98	0.5373	0.94	0.5339	0.86 (Z_H = 8.5)	0.7548
10.0	0.84	0.7419	−0.0319	0.94	0.5027	0.92	0.4989	0.82	0.6901
12.0	0.74	0.6751	−0.0286	0.80	0.4725	0.80	0.4685	0.84	0.6356
15.0	0.86	0.6162	−0.0229	0.72	0.4440	0.72	0.4411	–	–
18.0	0.66	0.5674	−0.0184	0.68	0.4169	0.70	0.4058	0.82	0.5739
20.0	0.66	0.5400	−0.0161	0.70	0.3847	0.70	0.3832	0.82 (Z_H = 19)	0.5524
30.0	0.64	0.4732	−0.00900	0.62	0.3383			0.64	0.4695
40.0	0.58	0.4387	−0.00575	0.56	0.3116			0.58	0.4365
50.0	0.54	0.4137	−0.00397	0.54	0.2973			0.52	0.4123
70.0	0.46	0.3765	−0.00230	0.52	0.2553			0.46	0.3758
100.0	0.40	0.3372	−0.00124	0.44	0.2264			0.40	0.3369

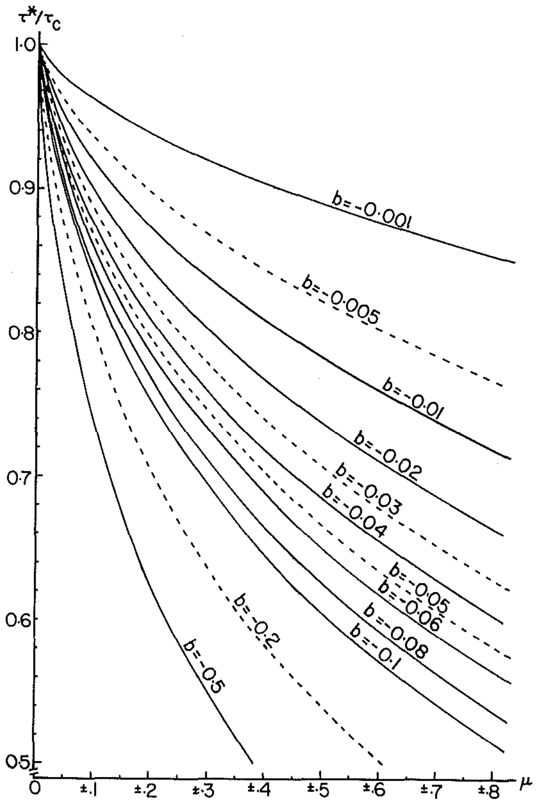

Fig. 1 Imperfection-sensitivity curves for various b coefficients

The in-plane boundary conditions for the above four coupled ODE for the second order fields can be obtained directly from equations (25)–(28) by replacing the wave number N by $2N$ and w_c, w_s, f_c, f_s by w_A, w_B, f_A, f_B, respectively.

In order to examine the stability of the single-mode symmetric system at the onset of buckling, it is necessary to compute the postbuckling b coefficient (see Hutchinson and Amazigo, 1967). The structure has stable postbuckling behavior (hence imperfection-insensitive) if b is positive and it has unstable postbuckling (imperfection-sensitive) if b is negative. The degree of imperfection sensitivity is measured by the magnitude of the postbuckling b coefficient. The equilibrium paths are specified by

$$b\xi^3 + [1-(\tau^*/\tau_c)]\xi = \mu(\tau/\tau_c) \quad (42)$$

where τ^* is the buckling load of the imperfect system, τ_c is the classical buckling load of the perfect system, and μ is the imperfection amplitude (assumed to be of the same shape as buckling mode) normalized with respect to the total thickness of the laminate. The imperfection amplitude is related to the buckling load (valid only if $b < 0$),

$$(3/2)(-3b)^{1/2}\mu(\tau/\tau_c) = [1-(\tau/\tau_c)]^{3/2} \quad (43)$$

The b coefficient is defined to be (Budiansky, 1967, and Hui, 1986b),

$$b = (C_1 + C_2)/|D_1| \quad (44)$$

where,

$$C_1 = 2\int_{y=0}^{y_o}\int_{x=0}^{Z_H} \{f_{\mathrm{I},yy} w_{\mathrm{I},x} w_{\mathrm{II},x} + f_{\mathrm{I},xx} w_{\mathrm{I},y} w_{\mathrm{II},y} - f_{\mathrm{I},xy}(w_{\mathrm{I},x} w_{\mathrm{II},y} + w_{\mathrm{I},y} w_{\mathrm{II},x})\}\,dx\,dy \quad (45)$$

$$C_2 = \int_{y=0}^{y_o}\int_{x=0}^{Z_H} \{f_{\mathrm{II},yy}(w_{\mathrm{I},x})^2 + f_{\mathrm{II},xx}(w_{\mathrm{I},y})^2 - 2f_{\mathrm{II},xy} w_{\mathrm{I},x} w_{\mathrm{I},y}\}\,dx\,dy \quad (46)$$

$$D_1 = (2\tau)\int_{y=0}^{y_o}\int_{x=0}^{Z_H} w_{\mathrm{I},x} w_{\mathrm{I},y}\,dx\,dy = (\tau N)y_o\int_{x=0}^{Z_H}(-w_c w_{s,x} + w_s w_{c,x})\,dx \quad (47)$$

In the above, $y_o = 2\pi R/(Rh)^{1/2}$ and the reduced-Batdorf parameter is defined to be the ratio of the shell length L to the characteristic length $(Rh)^{1/2}$, that is,

$$Z_H = L/(Rh)^{1/2} \quad \text{or} \quad Z_H = (L/R)(R/h)^{1/2} \quad (48)$$

so that the problem is independent of the individual values of L/R and R/h. Finally, substituting the buckling mode w_{I}, f_{I} and the second order fields w_{II} and f_{II} into C_1 and C_2, and performing the integration in the circumferential direction analytically, one obtains,

$$C_1 = (N^2 y_o)\int_{x=0}^{Z_H} \{w^*_{,x}(-f_s w_{s,x} - f_c w_{c,x}) + (1/2)w_{A,x}(f_s w_{s,x} - f_c w_{c,x}) - (1/2)w_{B,x}(f_s w_{c,x} + f_c w_{s,x}) - w_A(f_{s,xx} w_s - f_{c,xx} w_c) + w_B(f_{s,xx} w_c + f_{c,xx} w_s) + w_A(f_{s,x} w_{s,x} - f_{c,x} w_{c,x})$$

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$$\begin{aligned} &- w_B(f_{s,x}w_{c,x} + f_{c,x}w_{s,x}) - [w^*_{,x}(f_{s,x}w_s + f_{c,x}w_c) \\ &+ (1/2)w_{A,x}(f_{s,x}w_s - f_{c,x}w_c) \\ &+ (1/2)w_{B,x}(-f_{s,x}w_c - f_{c,x}w_s)]\}dx \end{aligned} \quad (49)$$

$$\begin{aligned} C_2 = (N^2 y_o)\int_{x=0}^{Z_H} &\{-f_A[-(w_{s,x})^2 + (w_{c,x})^2] \\ &- 2f_B w_{s,x}w_{c,x} + (1/2)f^*_{,xx}(w_s^2 + w_c^2) \\ &+ (1/4)f_{A,xx}(w_s^2 - w_c^2) - (1/2)f_{B,xx}w_s w_c \\ &+ f_{A,x}(w_s w_{s,x} - w_c w_{c,x}) \\ &- f_{B,x}(w_c w_{s,x} + w_s w_{c,x})\}dx \end{aligned} \quad (50)$$

4 Discussion of Results

Since it is impractical to present a complete parameter variation of the composite cylindrical shells, example problems are chosen from antisymmetric laminated cross-ply cylindrical shells made of Boron-epoxy materials. The material parameters are (Hui, 1985a,b),

$$E_1/E_2 = 10.0, \quad G_{12}/E_2 = 0.5, \quad \nu_{12} = 0.25$$

The in-plane boundary conditions in all the example problems are $U,_{YY}(X = 0) = 0$ and $V,_Y(X = 0) = 0$ with identical conditions at the other end $X = L$. The following four cases are being considered:

(*i*) Clamped, N_L = infinite,
(*ii*) Clamped, $N_L = 2$, 0 deg inside and 90 deg outside
(*iii*) Clamped, $N_L = 2$, 90 deg inside and 0 deg outside
(*iv*) Simply-supported, N_L = infinite

where N_L is the number of layers. Within the limits of where the Donnell shell theory is applicable (number of circumferential full-waves n is either 0 or greater than 5), the results presented are valid for all radius-to-thickness ratios. Thus, the geometry of the cylindrical shell can be described by the reduced-Batdorf parameter Z_H.

The coupled ordinary differential equations are discretized in the axial direction using the central finite difference scheme from $X = 0$ to $X = L$. The boundary conditions at each end of the shell are also discretized. For a given value of the circumferential wave number N, the smallest eigenvalue is obtained by using the shifted inverse power method. The classical torsional buckling load is obtained by repeating the procedure for all possible values of N. In all the present example problems, it is found that the torsional buckling load occurs in equal and opposite pairs, indicating that it is independent of the direction in which the torque is applied. This situation is quite analogous to the shear buckling of antisymmetric cross-ply rectangular plates (Hui, 1984) and cylindrical panels (Hui, 1985b). For other types of lamination, it must be noted that the direction of torque application may play an important role in laminated cylindrical shells (Marlowe et al., 1973, and Wilkins and Love, 1974) and shear buckling of laminated cylindrical panels (Zhang and Matthews, 1983). In the computation of the postbuckling b coefficient, the maximum deflection $(w_s{}^2 + w_c{}^2)^{1/2}$ is taken to be unity and it does not necessarily occur at the middle of the shell $X = L/2$. Integration in the axial direction is performed using Simpson's rule. It is important to point out that the present Koiter (1945) postbuckling analysis is valid only for imperfection shape identical to that of the buckling mode and that the imperfection amplitude should be sufficiently small.

Table 1 shows the torsional buckling data for each of the four cases of antisymmetric cross-ply cylindrical shells for various values of the reduced-Batdorf parameter Z_H. The optimal circumferential wave number N which corresponds to the classical torsional buckling load is also displayed and this decreases with increasing Z_H. It is found that the classical

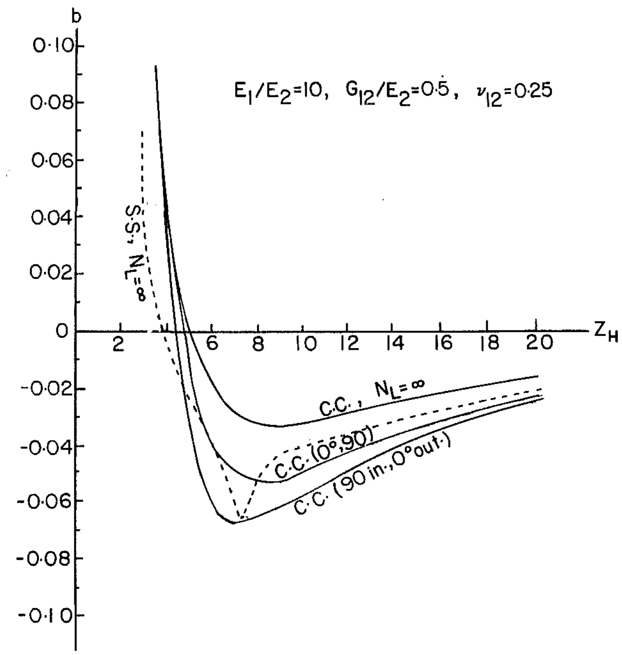

Fig. 2 Postbuckling *b* coefficient versus the reduced-Batdorf parameters for antisymmetric cross-ply Boron-epoxy cylinders under torsion

buckling load is not very sensitive to a slight change in the circumferential wave number, whereas the postbuckling b coefficient is quite sensitive to N. The classical buckling load decreases with increasing Z_H.

Figure 1 shows a graph of the buckling load (normalized with respect to the classical buckling load of the perfect system) versus the imperfection amplitude for various values of the b coefficients. The structure is imperfection-sensitive if b is negative and the degree of imperfection-sensitivity increases with increasing magnitude of the b coefficient.

Figure 2 shows a graph of the postbuckling b coefficient versus the reduced-Batdorf parameter for each of the four cases of antisymmetric cross-ply cylindrical shells. The b coefficients are positive for sufficiently small values of Z_H and the transition between positive and negative values of b occurs at Z_H between 3.0 and 5.0 for each of the four cases considered. Furthermore, the range of values of the reduced-Batdorf parameter Z_H which corresponds to the maximum imperfection-sensitivity is 4.0 to 9.0. For larger values of Z_H, the b coefficients remain negative but approach zero in all four cases. Judging from the magnitude of the b coefficients, the (0 deg inside and 90 deg outside) two-layered cylindrical shell is less imperfection-sensitive than the (90 deg, 0 deg) design due to different bending-stretching coupling behavior; however, the classical torsional buckling loads are identical for these two configurations. As a check on the analysis, the classical buckling load and the b coefficients for the special case of isotropic homogeneous cylindrical shells agree very well with the ones presented by Budiansky (1967) for all values of the Batdorf parameter.

Figure 3 shows the classical torsional buckling load of perfect orthotropic (N_L = infinite) cross-ply cylindrical shells versus Young's moduli ratio, keeping $G_{12}/E_2 = 0.5$ and $\nu_{12} = 0.25$. The torsional buckling load for the clamped cylinder with reduced-Batdorf parameter Z_H being 4 is found to be significantly higher than that for $Z_H = 10$ or 20. Moreover, as shown in Fig. 4, the b coefficients for $Z_H = 4$ are generally much more positive than that for $Z_H = 10$ or $Z_H = 20$. Thus, the $Z_H = 4$ configuration is a much better design than $Z_H = 10$ or 20. For fixed Young's moduli ratio between 1 and 40, the

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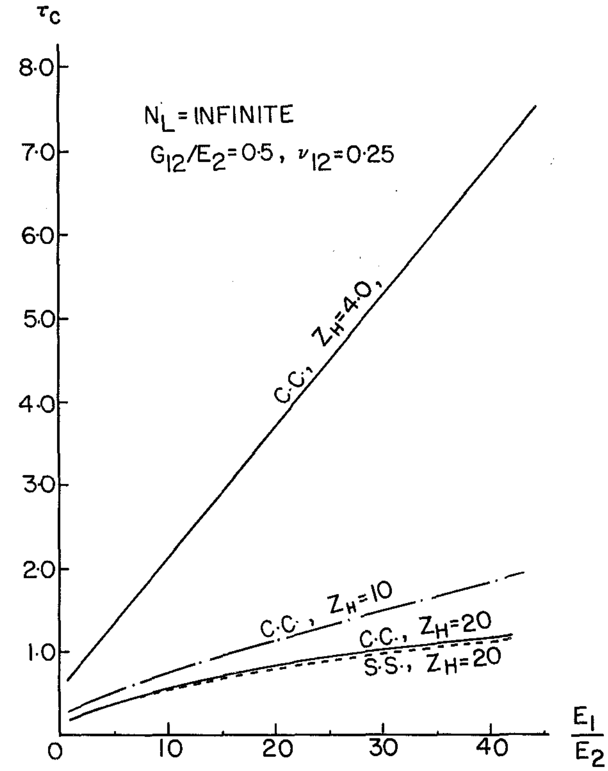

Fig. 3 Classical torsional buckling load versus Young's moduli ratio for orthotropic cross-ply cylindrical shells under torsion

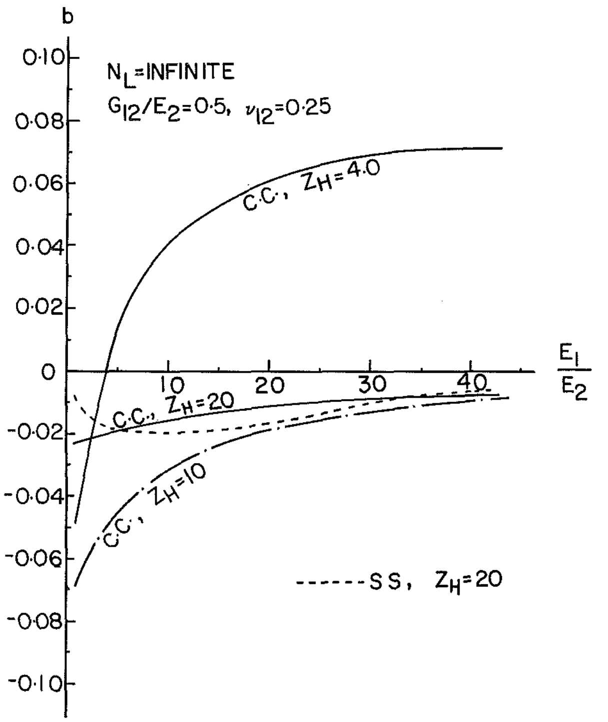

Fig. 4 Postbuckling *b* coefficient versus Young's moduli ratio for orthotropic cross-ply cylindrical shells under torsion

$Z_H = 10$ design has higher torsional buckling load than the $Z_H = 20$ configuration but at the expense of a more severe imperfection sensitivity. However, at least for fixed values of the reduced-Batdorf parameters being 4, 10, and 20, it appears that an increase in the Young's moduli ratio (causing a higher classical torsional buckling load) is not necessarily accompanied by an increase in the imperfection-sensitivity. This indicates that from the point of view of high buckling load and low imperfection-sensitivity, it is advantageous to use Graphite-epoxy ($E_1/E_2 = 40$) rather than Boron-epoxy ($E_1/E_2 = 10$) or Glass-epoxy ($E_1/E_2 = 3$) for cross-ply cylindrical shells under torsion.

Since the shape of the geometric imperfection considered by Simitses is not the same as that for the torsional buckling mode, a direct comparison between the present result with that of Simitses cannot be made. The present parameters involving the reduced-Batdorf parameter are in qualitative agreement with the imperfection sensitivity results of Simitses. However, the stronger geometries are not necessarily more sensitive to imperfection than the weaker ones as indicated by the present parameter variation involving Young's moduli ratio; this phenomenon is not supported in Simitses' findings, due to different material parameters and stacking sequences being considered.

Finally, the cross-ply cylindrical shell is most sensitive to imperfections (as indicated by relatively large negative values of b) in a rather low range of Z_H. That is, the shorter cylinders are more sensitive to imperfections than the longer ones. Since thin shell theory holds for R/h 30 (or so), the analysis is valid for sufficiently large values of Z_H. For example, if $L/R = 1$, $Z_H > 5.5$ and, similarly, if $L/R = 2$, $Z_H > 11$ and if $L/R = 10$, $Z_H > 55$.

5 Conclusions

The initial postbuckling behavior of antisymmetrically laminated cross-ply cylindrical shells under torsion has been examined. The torsional buckling loads and the associated postbuckling stiffness coefficients have been presented for Boron-epoxy cylindrical shells and concise parameter variations have been performed on the reduced-Batdorf parameter and Young's moduli ratio. Due to different bending-stretching coupling characteristics, the (0 deg inside and 90 deg outside) two-layer design is less imperfection-sensitive than the (90 deg, 0 deg) configuration. The laminated cylinders under torsion are imperfection-sensitive for the reduced-Batdorf parameter being approximately $4 \leq Z_H \leq 20$. Further, the increase in the torsional buckling load due to a higher value of Young's modulus ratio is not necessarily accompanied by a higher degree of imperfection-sensitivity.

It should be emphasized that these conclusions are valid only for certain shell configurations and boundary conditions. They are valid if at least three assumptions hold true: (*i*) The geometric imperfection is of the same shape as the torsional buckling mode (Koiter, 1945) and the imperfection amplitude is sufficiently small; (*ii*) the thin-shell theory is valid; (*iii*) membrane prebuckling state. For antisymmetric cross-ply layups, the last assumption is valid since pure shear is compatible (equation (19)) with a momentless prebuckling state with $b^*_{61} = b^*_{62} = b^*_{66} = 0$. It should be cautioned that a momentless prebuckling state is not consistent with the constitutive relations for antisymmetric "angle-ply" laminated cylindrical shells subjected to pure torsion. It is expected that angle-plies can provide greater torsional stiffness and strength and an extension of the present work to these laminates, which would involve a nonlinear prebuckling state (see Hui, 1986a), is in progress.

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