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POSTBUCKLING BEHAVIOR OF INFINITE BEAMS ON ELASTIC FOUNDATIONS USING KOITER'S IMPROVED THEORY

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Abstract—This study deals with the postbuckling behavior of infinite beams on non-linear elastic foundations subjected to axial compression. The analysis utilizes an improved Koiter's postbuckling theory such that the postbuckling coefficients are evaluated at the actual applied load rather than at the classical buckling load. The improved postbuckling paths are found to agree well with the non-linear large deflection solution using the Ritz procedure. This paper substantiates Koiter's conjecture that the general theory of elastic stability may be improved. The implications of various lower-bound buckling loads are examined.

1. INTRODUCTION

Buckling of beams on an elastic foundation has been examined by a number of authors (Hetenyi [1], Hunt [2], Amazigo *et al.* [3,4], Keener [5], Massalas and Tzivanidis [6] and Hui and Hansen [7]). Applications have been found in many civil engineering structures such as railway tracks (Kerr [8], Tvergaard and Needleman [9]), rocks and concrete pavements on soils etc. Due to recent interests in offshore technology and structural design in lake and ocean engineering, quite a few interesting papers can be found on buckling of sea or fresh-water semi-infinite floating ice sheets (Sodhi and Hamza [10], Wang [11]) and buckling of wedge-shaped floating ice sheets (Kerr [12] and Nevel [13]). Viscoelastic buckling of floating ice beams was studied by Sjolind [14] and an experimental determination of buckling loads of cracked ice sheets was carried out by Adley [15].

However, the influence of the higher order quartic terms of the potential energy (see Koiter [16]) on the postbuckling behavior of a beam on a non-linear elastic foundation has not been thoroughly analysed. It will be shown that the inclusion of all these higher order terms is essential for an accurate postbuckling analysis. The present paper is motivated by Koiter's conjecture, (see the concluding remarks of [17]), that "A better accuracy, however, may be achieved for larger values of $|\lambda_1 - 1|$ and $|\lambda - 1|$, if each of the coefficients C_1 and C_2 (that is, the b coefficients defined by Budiansky and Hutchinson [18]) is evaluated at the actual load factor λ, although we are unable to estimate the extended range of validity; we recommend to evaluate both C_1 and C_2 at the actual values of the load factor in a systematic numerical evaluation of the theory". Thus, a more accurate "improved" postbuckling analysis can be achieved by evaluating the postbuckling b coefficients at the actual applied load rather than at the classical buckling load of the perfect system. Except the author's work on cylindrical panels [19], it appears that Koiter's proposal has not yet been applied to any single-mode structural stability problem.

In this paper, the implications of both the usual and the improved b coefficients on the critical load and the postbuckling paths of elastically supported infinite beams under compression are examined. The improved solutions agree well with the non-linear large deflection analysis obtained from a two-term Galerkin procedure. Thus, the present findings substantiate Koiter's conjecture that the general theory of elastic stability can be improved. The present beam problem exhibits imperfection sensitivity buckling behavior similar to that of some thin shell problems; however, it will allow one to understand the complex postbuckling behavior unobscured by the much more complicated mathematical equations of thin-walled structures. Applications of the improved theory to mode interaction of

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axially stiffened cylindrical shells under compression will be the subject of a separate paper.

At present, there exist well over 25 papers which deal with the computation of the postbuckling b coefficients of various elastic single-mode initial postbuckling problems, all of which evaluate b at the classical buckling load ($\sigma = \sigma_c$). For example, such a simplified procedure is found in the overall smeared-out initial postbuckling analysis of ring and stringer reinforced cylindrical shells under axial compression and hydrostatic pressure (Hutchinson and Amazigo [20]), local panel buckling and initial postbuckling of stringer-reinforced cylindrical shells under axial compression (Koiter [21]), oval or elliptical cylindrical shells under axial compression (Hutchinson [22]), long or short cylindrical shells under lateral or hydrostatic pressure (Budiansky and Amazigo [23]) and torsion (Budiansky [24]). Excellent summaries of the above and the postbuckling behavior of many other thin-walled structures can be found in the survey papers by Hutchinson and Koiter [25] and Budiansky and Hutchinson [26]. Although the approximation of evaluating the b coefficients at $\sigma = \sigma_c$ is asymptotically correct, this paper aims to show that significant reduction in the degree of imperfection sensitivity can be obtained from a more accurate improved analysis. Thus, the usual simplifying procedure should be done only if the complications due to the algebra or numerical analysis are formidable.

In passing, it should be mentioned that the application of the improved theory to multi-mode postbuckling analysis may be quite laborious especially in the case of an externally pressurized spherical shell at which there is a whole cluster of modes with eigenvalues just above the critical smallest eigenvalue (Koiter [27]). The severe imperfection sensitivity reported in [27] is primarily due to the rapidly varying interaction with all these modes; thus, it is not clear to what extent the degree of imperfection sensitivity of a single-mode system (without mode interaction) may be affected by employing the improved theory. It appears that a comparison between the imperfection sensitivity results obtained by Koiter's usual and improved theories has not been reported in the open literature. An estimation of the range of validity of the asymptotic method to predict mode interaction of the Van der Neut columns was presented by Byskov [28].

2. POTENTIAL ENERGY AND THE CLASSICAL BUCKLING LOAD

The potential energy of a beam (the neutral axis is assumed to be incompressible) on a non-linear elastic foundation subjected to a concentrated axial compressive force P (positive for compression) can be written in the form (Koiter [16] and Hunt [2]),

$$\text{P.E.} = (EI/2)\int_{X=0}^{L} \{(W_{,XX})^2/[1-(W_{,X})^2]\}\,\mathrm{d}X - P\int_{X=0}^{L}\{1-[1-(W_{,X})^2]^{1/2}\}\,\mathrm{d}X$$

$$+\int_{X=0}^{L}\{(K_1/2)(W^2)+(K_2/3)(W^3)+(K_3/4)(W^4)\}\,\mathrm{d}X. \tag{1}$$

In the above, E is Young's modulus, I is the moment of inertia of the beam, W is the lateral deflection, L is the length of the beam, X is the axial coordinate measured from $X = 0$ to $X = L$ and K_1 (unit force per square length), K_2 and K_3 are the linear, quadratic and cubic spring constants of the non-linear elastic foundation. The following non-dimensional quantities are introduced:

$$w = W/\lambda, \qquad x = X/\lambda, \qquad \sigma = P\lambda^2/(4EI) = P/[2(EIK_1)^{1/2}]$$

$$k_2 = K_2\lambda^5/(EI), \qquad k_3 = K_3\lambda^6/(EI), \qquad \lambda = (4EI/K_1)^{1/4} \tag{2}$$

where λ is the characteristic length of the elastically supported beam (Hetenyi [1]). Thus, the potential energy can be expressed according to the order of the lateral deflection in the non-dimensional form,

$$\text{P.E.} = \left(\frac{EI}{2\lambda}\right)(P_2^0[u] + P_2'[u] + P_3^0[u] + P_4^0[u] + P_4'[u]) \tag{3}$$

where,

$$P_2^0[u] = \int_{x=0}^{L/\lambda} \{(w_{,xx})^2 + 4w^2\}\,\mathrm{d}x$$

$$P_2'[u] = (-\sigma)\int_{x=0}^{L/\lambda} 4(w_{,x})^2\,\mathrm{d}x$$

$$P_3^0[u] = (2k_2/3)\int_{x=0}^{L/\lambda} w^3\,\mathrm{d}x \qquad \text{(4a–e)}$$

$$P_4^0[u] = \int_{x=0}^{L/\lambda} \{(w_{,xx})^2(w_{,x})^2 + (k_3/2)(w^4)\}\,\mathrm{d}x$$

$$P_4'[u] = (-\sigma)\int_{x=0}^{L/\lambda} (w_{,x})^4\,\mathrm{d}x.$$

It should be noted that unlike the conventional Koiter notation, the applied axial load σ is incorporated into the $P_2'[u]$ and $P_4'[u]$ expressions (Hui [29]).

By minimizing the quadratic terms of the potential energy, the governing differential equation for classical buckling is,

$$w_{c,xxxx} + (4\sigma)w_{c,xx} + 4w_c = 0. \qquad (5)$$

The beam is taken to be of sufficient length such that the boundary conditions at the two ends of the beam may be neglected in favor of the periodicity requirement of the lateral deflection. Assuming that buckling mode to be of the form,

$$w_c = (\xi/\lambda)\sin(Mx) \qquad (6)$$

where ξ is the amplitude of the buckling mode, the classical buckling load σ_c and the associated buckled wave number is found to be,

$$\sigma_c = 1, \qquad M^2 = 2. \qquad (7)$$

Since the quadratic and cubic spring constants k_2 and k_3 do not appear in the quadratic terms of the potential energy, they will not affect the buckling load and the buckling mode. However, their influence on the initial postbuckling behavior will be examined in the subsequent sections.

3. INITIAL POSTBUCKLING BEHAVIOR

According to Koiter–Budiansky–Hutchinson's postbuckling theory for single-mode structural system, the potential energy expanded at the classical buckling load can be written in the form

$$\text{P.E.} = (A_4)(\xi/\lambda)^4 + (\sigma - \sigma_c)(d_1)(\xi/\lambda)^2 + (2d_1\sigma)(\bar{\xi}/\lambda)(\xi/\lambda) \qquad (8)$$

where $\bar{\xi}$ is the amplitude of the initial geometric imperfection which is taken to be of the same shape as the buckling mode. Furthermore, $A_4(\xi/\lambda)^4$ and $d_1(\xi/\lambda)^2$ are defined to be (Hui [30]),

$$A_4(\xi/\lambda)^4 = P_4^0[u_c] + P_4'[u_c] - (P_2^0[u_{11}] + P_2'[u_{11}]) \qquad (9a)$$

$$d_1(\xi/\lambda)^2 = \frac{\mathrm{d}}{\mathrm{d}\sigma}P_2'[u_c] \qquad (9b)$$

where u_c is the buckling mode and u_{11} is the second order field displacement such that

$$w_{\text{total}} = (\xi/\lambda)w_c + (\xi/\lambda)^2 w_{11}. \tag{10}$$

The equilibrium path equation is obtained by minimizing the expanded potential energy respect to the normalized amplitude (ξ/λ), that is,

$$(b)(\xi/\lambda)^3 + [1 - (\sigma/\sigma_c)](\xi/\lambda) = (\sigma/\sigma_c)(\bar{\xi}/\lambda) \tag{11}$$

where the postbuckling b coefficient is defined to be,

$$b = 4A_4/|-2d\sigma_c|. \tag{12}$$

Further, from the second variation of the expanded potential energy with respect to the amplitude ξ/λ, the stability boundary is governed by,

$$(3b)(\xi/\lambda)^2 + 1 - (\sigma/\sigma_c) = 0. \tag{13}$$

In the special case of a perfect system ($\bar{\xi} = 0$), the equilibrium path is specified by,

$$\sigma/\sigma_c = 1 + b(\xi/\lambda)^2. \tag{14}$$

At this stage, it should be noted that if the b coefficient of a single-mode symmetric system is positive, then the initial postbuckling behavior is stable, otherwise, it is unstable. In the case where b is negative, a relationship between the critical load and the amplitude of the geometric imperfection can be obtained by eliminating the amplitude ξ/λ from the equilibrium path and the stability boundary equations to yield,

$$\frac{(2/3)[1 - (\sigma/\sigma_c)]^{3/2}}{(-3b)^{1/2}} = (\sigma/\sigma_c)(\bar{\xi}/\lambda). \tag{15}$$

In the present example of an infinite beam on a non-linear elastic foundation, the cubic term of the potential energy evaluated at the buckling mode is zero (regardless of the value of k_2) due to the periodic nature of the buckling mode of the infinite beam. Hence the a coefficient (Budiansky and Hutchinson [18]) vanishes. In order to compute the b coefficient, it is necessary to obtain the second order field displacement w_{11}. Using Koiter's notation that,

$$P_2[u + v] = P_2[u] + P_{11}[u, v] + P_2[v] \tag{16}$$

the governing differential equation for the second order field displacement is (Koiter [16,21].

$$P_{11}[u_c, \delta u_{11}] = (-1)P_{21}[u_c, \delta u_{11}]. \tag{17}$$

That is, for the present infinite beam problem,

$$2w_{11,xxxx} + 8w_{11} + 8\sigma w_{11,xx} = (-2k_2)(w_c)^2. \tag{18}$$

The second order field which satisfies the above differential equation is

$$w_{11} = (-k_2/8) + c_0 \cos(2Mx) \tag{19a}$$

where,

$$c_0 = \frac{k_2}{(8)(4M^4 - 4\sigma M^2 + 1)} = k_2/[(8)(17 - 8\sigma)]. \tag{19b}$$

Note that $k_2 = 0$ implies that the differential equation of the second order field w_{11} is identical to that of the buckling mode; thus, $w_{11} = 0$ by virtue of the orthogonal condition between w_{11} and w_c (Koiter [16]). Substituting w_c and w_{11} into the quartic terms of the potential energy, one obtains,

$$A_4(\xi/\lambda)^4 = \{1 - (3\sigma/2) + (3k_3/16) - (k_2)^2 g(\sigma)\}(L/\lambda)(\xi/\lambda)^4 \tag{20}$$

where,

$$g(\sigma) = \left\{1 + \left(\frac{1}{(2)(4M^4 - 4\sigma M^2 + 1)}\right)\right\}(1/16). \tag{21}$$

It can be seen that $g(\sigma = 1) = 0.06597222$ and $g(\sigma = 0) = 0.06490385$ so that the function $g(\sigma)$ is not sensitive to the applied load.

As a check on the analysis, the differential equation of the second order field [equation (17)] implies that $A_4(\xi/\lambda)^4$ can be written in the alternative form (Koiter [21]),

$$A_4(\xi/\lambda)^4 = P_4^0[u] + P_4'[u] + (1/2)P_{21}[u_c; u_{11}] \tag{22}$$

and it is found that (20) can also be obtained from (22). Further, the quantity d_1 can be evaluated to become,

$$d_1(\xi/\lambda)^2 = -4(L/\lambda)(\xi/\lambda)^2. \tag{23}$$

Thus, the b coefficient is found to be a function of the applied load in the form,

$$b(\text{improved}) = (1/2)[1 - (3\sigma/2)] + \bar{k} \tag{24}$$

where,

$$\bar{k} = (3k_3/32) - (k_2)^2(1/2)g(\sigma). \tag{25}$$

Since $g(\sigma)$ is not sensitive to the applied load, $\bar{k}$ may be approximated by,

$$\bar{k} = (3k_3/32) - 0.03298611(k_2)^2 \tag{26}$$

and the b coefficient evaluated at $\sigma = \sigma_c$ is,

$$b(\text{usual}) = (-1/4) + (3k_3/32) - 0.03298611(k_2)^2. \tag{27}$$

The b (usual) coefficient agrees with the author's earlier analysis on buckling of finite-length beam on elastic foundation (Hui and Hansen [7]) by letting $j\pi = (2)^{1/2}(L/\pi)^2$.

For an imperfect system, the applied load at which the b (improved) coefficient is zero is of interest because for applied loads below this value, the b coefficient evaluated at the actual load is positive so that equation (15) is no longer valid (that is, the structure cannot buckle). By setting b (improved) = 0, the lower bound buckling load is found to be

$$\text{L.B.} = (2/3) + (4\bar{k}/3). \tag{28}$$

On the other hand, such a lower bound buckling load exists if the b coefficient is evaluated at the classical buckling load. Moreover, one should be aware that this lower bound is subjected to the limitation that the asymptotic analysis is valid only for sufficiently small amplitude of the initial geometric imperfection. Thus, the terminology "lower bound" should be understood to mean "tentative lower bound". Nevertheless, it is quite certain that if the critical load of an imperfect system falls below the lower bound buckling load (L.B.) using the "usual" method of evaluating the b coefficient at the classical buckling load, then the usual result should be viewed with suspicion.

Finally, the equilibrium path equation can be obtained by substituting the b(improved) coefficient into equation (11) and upon grouping of terms involving the applied load, one obtains,

$$[\bar{k} + (1/2)](\xi/\lambda)^3 + (\xi/\lambda) = (\sigma/\sigma_c)[(\xi/\lambda) + (\bar{\xi}/\lambda) + (3/4)(\xi/\lambda)^3]. \tag{29}$$

The equation which governs the stability boundary is,

$$1 + [3\bar{k} + (3/2)](\xi/\lambda)^2 = (\sigma/\sigma_c)[1 + (9/4)(\xi/\lambda)^2]. \tag{30}$$

4. VERIFICATION OF KOITER'S IMPROVED POSTBUCKLING THEORY

One-term Ritz analysis

In order to assess the accuracy of Koiter's improved postbuckling theory in the case of an elastically supported infinite beam, the large deflection problem is solved using the Rayleigh–Ritz procedure. The deflection is assumed to be of the sinusoidal form,

$$w = A \sin(Mx). \tag{31}$$

Substituting this deflection into the potential energy equation (3) and evaluating the integral (taking into account that the beam is infinitely long in the integration), one obtains, for the perfect system,

$$\text{P.E.} = \left(\frac{EI}{2\lambda}\right)((A^2)\{(M^4/2) + 2 - 2\sigma M^2\} + (A^4)(1/8)\{M^6 + (3k_3/2) - 3\sigma M^4\}). \tag{32}$$

Minimizing the above potential energy with respect to A^2, one obtains, the equilibrium equation for the postbuckling path in the form,

$$\sigma = \left(\frac{2M^4 + 8 + (A^2)[M^6 + (3k_3/2)]}{8M^2 + 3M^4A^2}\right). \tag{33}$$

Since the wave number $M^2 = 2$ for the buckling mode, one obtains,

$$\sigma = \left(\frac{4 + (A^2)[2 + (3k_3/8)]}{4 + 3A^2}\right). \tag{34}$$

Thus, the above equilibrium equation agrees exactly with that obtained using the improved postbuckling theory [see equation (29) with $\bar{\xi} = 0$ and $k_2 = 0$]. In order to study the influence of k_2 on the postbuckling behavior (even though a non-zero value of k_2 may be physically unrealistic), a multi-mode Ritz solution is needed.

Two-term Ritz analysis

A more accurate study of the postbuckling problem can be achieved by assuming a two-term approximation of the lateral deflection in the Ritz procedure, that is,

$$w = A \sin(Mx) + B \sin^2(Mx). \tag{35}$$

Substituting the above lateral deflection into the potential energy expressions and evaluating the integral, one obtains,

$$P_2^0[u_c] + P_2'[u_c] = (A^2)[(M^4/2) + 2 - 2\sigma M^2] + (B^2)[2M^4 + (3/2) - 2\sigma M^2] \quad (36)$$

$$P_3[u_c] = (k_2)[(A^2B)(3/4) + (B^3)(5/24)] \quad (37)$$

$$\begin{aligned} P_4^0[u_c] + P_4'[u_c] &= (A^4)(M^6/8) + (B^4)(M^6/2) + (A^2B^2)(15M^6/8) \\ &+ (k_3/2)[A^4(3/8) + (A^2B^2)(15/8) + B^4(35/128)] \\ &- (\sigma)[(A^4)(3M^4/8) + (A^2B^2)(9M^4/4) + (B^4)(3M^4/8)]. \end{aligned} \quad (38)$$

Minimizing the total potential energy [equation (3)] with respect to the amplitudes A and B, one obtains two equilibrium equations of the form,

$$(A)[2r_1 + (3k_2/2)(B) + 4r_3(A^2) + 2r_4(B^2)] = 0 \quad (39)$$

$$2r_2B + (3k_2/4)(A^2) + (5k_2/8)(B^2) + 2r_4(A^2B) + 4r_5(B^3) = 0 \quad (40)$$

where

$$\begin{aligned} r_1 &= 4 - 4\sigma \\ r_2 &= (19/2) - 4\sigma \\ r_3 &= 1 + (3k_3/16) - (3\sigma/2) \\ r_4 &= 15 + (15k_3/16) - 9\sigma \\ r_5 &= 4 + (35k_3/256) - (3\sigma/2). \end{aligned} \quad (41)$$

Eliminating A^2 from equations (39–40) yields a cubic equation in the amplitude B (which can be solved in closed form),

$$\begin{aligned} &[4r_5 - (r_4^2/r_3)](B^3) + (k_2)[(5/8) - (9/8)(r_4/r_3)](B^2) \\ &\quad + [2r_2 - (r_1r_4/r_3) - (9/32)(k_2^2/r_3)](B) - (3k_2/8)(r_1/r_3) = 0. \end{aligned} \quad (42)$$

Once the amplitude B is found, the amplitude A can then be easily obtained from equation (39).

5. DISCUSSION OF RESULTS

Figure 2 shows a graph of the b(improved) and b(usual) coefficients vs the non-dimensional cubic spring constant k_3 with $k_2 = 0$. It can be seen that in the important special case of a linear elastic foundation alone ($k_2 = k_3 = 0$), b(usual) is found to be -0.25 and thus, the corresponding postbuckling behavior is unstable. However, stable

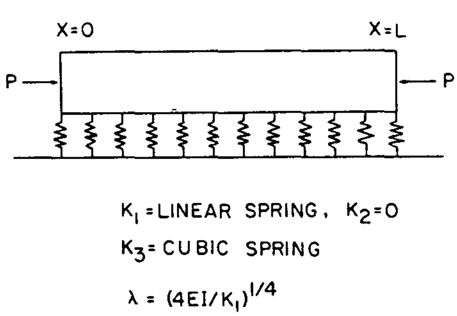

Fig. 1. A beam on a non-linear elastic foundation under axial compression.

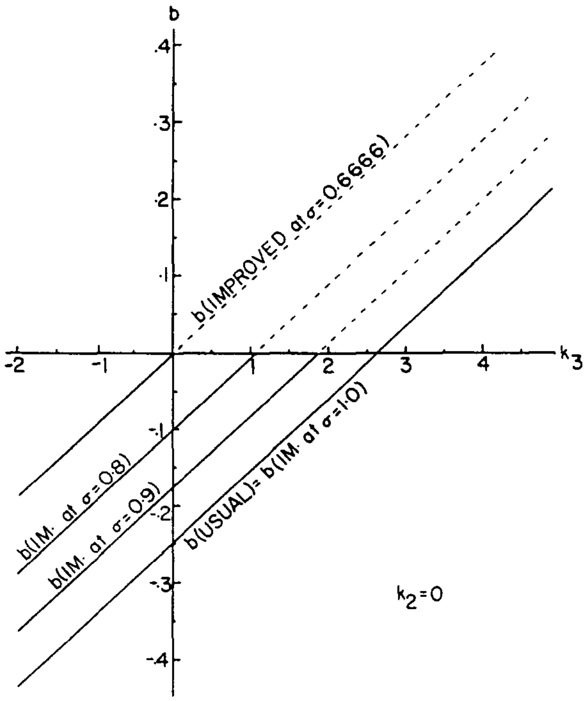

Fig. 2. The b(improved) coefficient vs the non-dimensional cubic spring constant for various values of the applied load ($k_2 = 0$).

postbuckling behavior can be obtained by introducing a sufficiently large value of the cubic spring constant $k_3 > 2.6666$ while keeping $k_2 = 0$. The b(improved) coefficient for various values of the applied load being $\sigma = 0.9$, $\sigma = 0.8$ and $\sigma = 2/3$ show a significant positive shift of the b coefficients, thus, a substantial reduction of the degree of imperfection sensitivity. Due to the asymptotic character of the problem, the dotted lines (which indicate sign changes of the b coefficient) should be viewed with reservation.

The critical load vs amplitude of the normalized geometric imperfection ($\bar{\xi}/\lambda$) for the special case when $k_2 = 0$ is shown in Fig. 3 for two values of k_3 being 0 and 2. It can be seen that the imperfection-sensitivity curves based on the b(usual) and b(improved) coefficients coincide asymptotically. However, a substantial difference between the usual and improved curves exists in the critical loads for $k_3 = 2$ since the b(usual) is a small negative quantity being -0.0625, implying a positive shift in the b coefficient may change its sign and thus, have a pronounced effect on the critical load versus imperfection amplitude curve. Furthermore, the improved curve flattens out rapidly and the tentative lower bound is found to be 0.9166. On the other hand, a relatively less pronounced difference between the usual and improved curves is seen in the case $k_3 = 0$ since the b(usual) coefficient is already a relatively large negative quantity being -0.25. Moreover, the improved curve flattens out less rapidly and the lower bound buckling load is L.B. = 0.6666.

The tentative lower bound calculation may be quite useful for predicting the range in the critical load–imperfection curve at which the usual curve is no longer valid. For example, the usual curve for $k_3 = 2$ in Fig. 3 is not valid for the portion of the curve where the critical load falls below 0.9166. It is possible that the region of validity of the improved curve may be somewhat larger than the usual curve. An estimate of the "extended" range of validity can be made by comparing the improved solution with the non-linear large deflection postbuckling curves obtained from a two-term Rayleigh–Ritz procedure.

Figure 4 shows the postbuckling path of elastically supported infinite beams on elastic foundation with no geometric imperfection. Again, the postbuckling paths obtained by using the b(usual) and b(improved) coefficients coincide asymptotically (that is, for sufficiently small amplitude (ξ/λ)). However, the improved and the usual curves differ significantly for larger values of ξ/λ. In fact, the improved post-buckling paths tend to

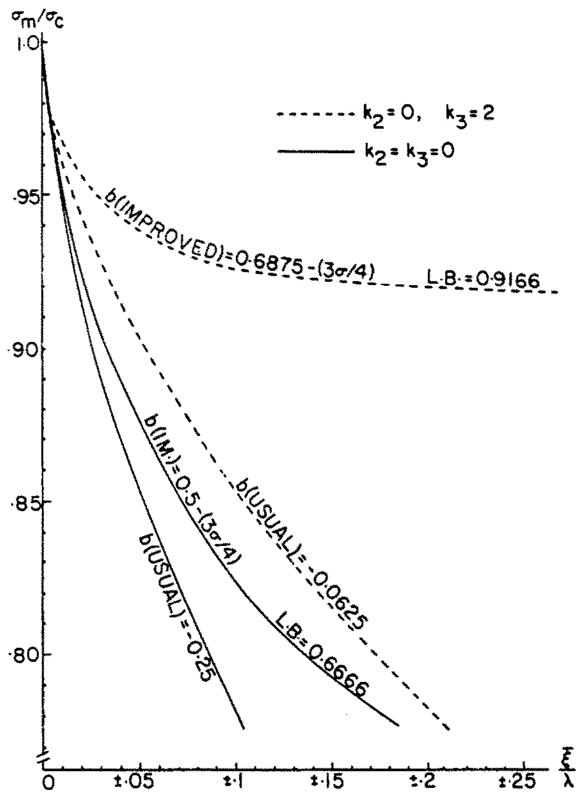

Fig. 3. The critical load vs the normalized imperfection amplitude using the b(usual) and b(improved) coefficients (for $k_2 = k_3 = 0$ and $k_2 = 0$, $k_3 = 2$).

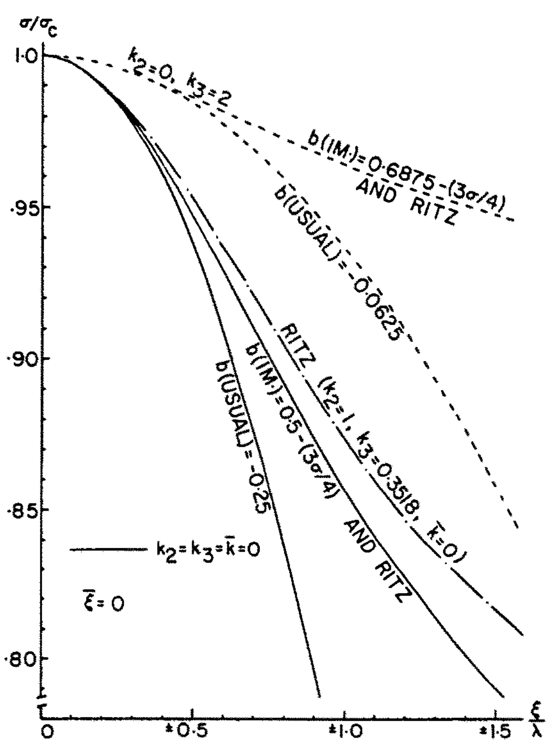

Fig. 4. Postbuckling paths of imperfect infinite beams using the b(usual) and b(improved) coefficients ($k_2 = k_3 = 0$ and $k_2 = 0$, $k_3 = 2$).

flatten out and tend to a lower bound value of 11/12 (that is, 0.9166) for ($k_2 = 0$) and $k_3 = 2$) and 0.6666 for ($k_2 = k_3 = 0$). No such lower bound load can be found for the usual postbuckling curves.

Using a two-term Ritz procedure, it is found that the amplitude B is at most one order of magnitude smaller than A. The postbuckling paths of applied load vs amplitude A using the Ritz procedure agree very well (too closed to be plotted) with that predicted by Koiter's improved postbuckling theory. It appears that the range of validity of the improved curves may indeed be larger than that for the usual curves. Both the Ritz procedure and Koiter's improved theory predict the existence of lower bounds whereas no such lower bound can be found using Koiter's 1945 theory. It appears that for non-zero values of k_2, a larger number of terms in the Ritz procedure may be necessary for an accurate postbuckling analysis. In view of the unrealistic nature of $k_2 \neq 0$, a more accurate Ritz procedure will not be presented in this paper.

Finally, a symmetric single-mode structural system in which the b coefficient is a small negative quantity is found quite frequently in many shell instability problems (Koiter [17, 21] Hutchinson and Koiter [25]). Of particular concern is the local and overall mode interaction of axially stiffened cylindrical shells under compression (Koiter [17, 21]) in which the b coefficients may be quite sensitive to the applied load. In all the above stability problems on shells, it was found that the quartic term of the potential energy is independent of the applied load, that is, $P_4'[u] = 0$. Further, $P_2'[u]$ and hence d_1 is always a negative quantity (otherwise, the structure cannot buckle). Consequently, the evaluation of the postbuckling b coefficient at the actual applied load σ rather than at $\sigma = \sigma_c$ implies that A_4 and hence the b coefficient will become more positive (that is, the shift is always positive in the b coefficient). Thus, the imperfection sensitivity behavior predicted by the usual Koiter–Budiansky–Hutchinson general postbuckling theory is overestimated. The present investigation may have significant practical implications since virtually all the postbuckling analyses which employed Koiter's general theory of 1945 reported in the open literature have overestimated the imperfection sensitivity of thin-walled structures.

Finally, it should be mentioned that the improved postbuckling analysis and Koiter's amplitude modulation theory (Koiter [17] and Koiter and Pignataro [31]) seem to produce similar trends in that there exist lower bound buckling loads and the imperfection sensitivity is less serious than that predicted by Koiter's theory of 1945.

6. CONCLUDING REMARKS

The buckling and postbuckling behavior of infinite beams on non-linear elastic foundations under axial compression have been analyzed. It is found that the postbuckling behavior in the case of a linear elastic foundation alone is unstable. The imperfection sensitivity and the postbuckling paths are plotted based on both Koiter's general theory and Koiter's improved postbuckling theory. The improved solution agrees with the non-linear large deflection analysis and it may differ significantly with the usual solution. The usefulness of the improved postbuckling theory are discussed in terms of the lower bound buckling load and it is observed that the usual theory overestimates the imperfection sensitivity in the finite deflection regime.

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